Computation Appendix

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1 Introduction

This note describes how to compute a heterogenous trading economy with solvency constraints. Please refer to Chien, Cole and Lustig (2008) for the details of this economy.

1.1 The Stationary Analog

Our economy is non-stationary because of our assumption that the growth shocks have permanent effects through $Y(z^t)$. In order to compute an equilibrium of our economy we will construct a stationary analog and show how to map the stationary equilibrium objects into an equilibrium for our non-stationary economy and vice versa. This involves showing that there is a simple change of variables and a new utility index that yields a stationary analog to our model economy. In order to construct the stationary analog, we will make the following assumption.

Assumption 1.1 *Let*

$$\widehat{\pi}(z_{t+1}|z_t) = \frac{\pi(z_{t+1}|z_t)(\exp\{z_{t+1}\})^{1-\alpha}}{\sum_{z_{t+1}} \pi(z_{t+1}|z_t)(\exp\{z_{t+1}\})^{1-\alpha}}, \tag{1}$$

$$\widehat{\beta}(z_t) = \beta \sum_{z_{t+1}} \pi(z_{t+1}|z_t)(\exp\{z_{t+1}\})^{1-\alpha}, \tag{2}$$

$$\frac{\hat{P}(z^{t+1})}{\hat{P}(z^t)} = \beta^{-1}\widehat{\beta}(z_t)\exp(z_{t+1})\frac{P(z^{t+1})}{P(z^t)}, \tag{3}$$

$$\hat{c}(z^t, \eta^t) = \frac{c(z^t, \eta^t)}{Y(z^t)}, \tag{4}$$

$$\widehat{S}(\hat{\zeta}(z^t,\eta^t);z^t,\eta^t) = \frac{S(\zeta(z^t,\eta^t);z^t,\eta^t)}{Y(z^t)}, \tag{5}$$

Define the new utility index $\hat{U}(\cdot)$ *as*

$$\hat{U}(\widehat{c})(z^t,\eta_t) = u(\widehat{c}_t(s^t)) + \widehat{\beta}(z_t) \sum_{z_{t+1},\eta_{t+1}} \widehat{\pi}(z_{t+1}|z_t)\pi(\eta_{t+1}|z_{t+1},\eta_t)\hat{U}(\widehat{c})(z^{t+1},\eta^{t+1}). \tag{6}$$

With respect to our multipliers, let

$$\begin{aligned} \hat{\zeta}(z^t,\eta^t) &= \zeta(z^t,\eta^t),\ \hat{\nu}\left(z^t,\eta^t\right) = \nu(z^t,\eta^t),\ \textit{and} \\ \hat{\varphi}\left(z^t,\eta^t\right) &= \varphi(z^t,\eta^t). \end{aligned} \tag{7}$$

There is a difference between the two economies. The flow budget constraint in the growth deflated economy is given by

$$\begin{aligned} &\gamma\eta_t + \widehat{a}(z^{t-1},\eta^{t-1}) + \sigma(z^{t-1},\eta^{t-1})\left[(1-\gamma) + \widehat{\varpi}(z^t)\right] \\ \geq\ &\widehat{c}(z^t,\eta^t) + \sum_{z^{t+1}|z^t} q(z_{t+1},z^t) \sum_{\eta^{t+1}\succ\eta^t} \widehat{a}(z^{t+1},\eta^{t+1})(\eta^{t+1}|z^{t+1},\eta^t) + \sigma(z^t,\eta^t)\widehat{\varpi}(z^t)\ \forall t. \end{aligned} \tag{8}$$

The stationary net savings recursion is given by

$$\begin{aligned} \widehat{S}(\hat{\zeta}(z^t,\eta^t);z^t,\eta^t) &= \left[\gamma\eta(z^t,\eta^t) - \widehat{c}(z^t,\eta^t)\right] + \\ & \sum_{z^{t+1},\eta^{t+1}} \hat{\pi}(z^{t+1},\eta^{t+1}|z^t,\eta^t)\exp(z_{t+1})^{\gamma-1}\frac{\hat{P}(z^{t+1})}{\hat{P}(z^t)}\widehat{S}(\hat{\zeta}(z^{t+1},\eta^{t+1});z^{t+1},\eta^{t+1}), \end{aligned} \tag{9}$$

and the debt bound in the stationary economy is given by

$$\widehat{S}(\hat{\zeta}(z^{t+1},\eta^{t+1});z^{t+1},\eta^{t+1}) \leq \frac{\underline{M}(z^{t+1},\eta^{t+1})}{Y(z^{t+1})}. \tag{10}$$

Proposition 1.1 *For any equilibrium of our time-zero trading economy* $\{P_t(z^t),\ q_t(z^t),\ \varpi_t(z^t),\ c_t(z^t,\eta^t),\ Y(z^t)\}$ *with multipliers* $\{\zeta(z^t,\eta^t),\nu(z^t,\eta^t),\varphi(z^t,\eta^t)\}$*, the associated analog defined by equations (1)-(7) with revised constraints (8) and (10) is an equilibrium for the stationary economy and vice versa.*

Proof. Utility: First we show that households rank consumption streams $\{c_t(z^t,\eta^t)\}$ in the original economy in exactly the same way as they rank growth-deflated consumption streams $\{\hat{c}_t(z^t,\eta^t)\}$.

Denote $U(c)(z^t,\eta^t)$ as continuation utility of an agent from consumption stream c, starting at history (z^t,η^t). This continuation utility follows the simple recursion

$$U(c)(z^t,\eta^t) = u(c_t(z^t,\eta^t)) + \beta \sum_{(z^{t+1},\eta^{t+1})} \pi(z^{t+1},\eta^{t+1}|z^t,\eta^t) U\left[c((z^t,\eta^t),z_{t+1},\eta_{t+1})\right]$$

Divide both sides by $Y(z^t)^{1-\gamma}$ to obtain

$$\begin{aligned}
\frac{U(c)(z^t,\eta^t)}{Y(z^t)^{1-\gamma}} &= u(\widehat{c}_t(z^t,\eta^t)) \\
&\quad +\beta \sum_{z_{t+1},\eta_{t+1}} \pi(z_{t+1}|z_t)\pi(\eta_{t+1}|z_{t+1},\eta_t) \frac{Y(z^{t+1})^{1-\gamma}}{Y(z^t)^{1-\gamma}} \frac{U\left[c((z^t,\eta^t),z_{t+1},\eta_{t+1}\right]}{Y(z^{t+1})^{1-\gamma}} \\
&= u(\widehat{c}_t(z^t,\eta^t)) \\
&\quad +\hat{\beta}(z_t) \sum_{z_{t+1},\eta_{t+1}} \hat{\pi}(z_{t+1}|z_t)\pi(\eta_{t+1}|z_{t+1},\eta_t) \frac{U\left[c((z^t,\eta^t),z_{t+1},\eta_{t+1}\right]}{Y(z^{t+1})^{1-\gamma}},
\end{aligned}$$

and it follows that

$$\frac{U(c)(z^t,\eta^t)}{Y(z^t)^{1-\gamma}} = u(\hat{c}_t(s^t)) + \widehat{\beta}(s_t) \sum_{s_{t+1}} \widehat{\pi}(s_{t+1}|s_t) \hat{U}(\hat{c}^i)(s^t,s_{t+1}) = \hat{U}(\hat{c})(s^t).$$

Hence we order original and growth-deflated consumption streams in exactly the same way.

Pricing: Second, we want to show that objects are priced similarly in the two economies. Note that if

$$S(z^t,\eta^t) = \left[\eta(z^t,\eta^t)Y(z^t) - c(z^t,\eta^t)\right] + \sum_{z^{t+1},\eta^{t+1}} \frac{\pi(z^{t+1},\eta^{t+1})P(z^{t+1})}{\pi(z^t,\eta^t)P(z^t)} S(z^{t+1},\eta^{t+1}),$$

then

$$\begin{aligned}
\frac{S(z^t,\eta^t)}{Y(z^t)} &= \left[\frac{\eta(z^t,\eta^t)Y(z^t) - c(z^t,\eta^t)}{Y(z^t)}\right] + \sum_{z^{t+1},\eta^{t+1}} \frac{\widehat{\pi}(z_{t+1}|z_t)}{\exp(z_{t+1})} \frac{\hat{P}(z^{t+1})}{\hat{P}(z^t)} \frac{D(z^{t+1},\eta^{t+1})}{Y(z^t)} \\
&= \left[\eta(z^t,\eta^t) - \hat{c}(z^t,\eta^t)\right] + \\
&\quad \sum_{z^{t+1},\eta^{t+1}} \hat{\pi}(z^{t+1},\eta^{t+1}|z^t,\eta^t)\exp(z_{t+1})^{\gamma-1} \frac{\hat{P}(z^{t+1})}{\hat{P}(z^t)} \frac{S(z^{t+1},\eta^{t+1})}{Y(z^{t+1})},
\end{aligned}$$

which implies the same recursion in our growth-deflated economy given the definition of $\widehat{S}(z^t,\eta^t)$.

Debt Bound: Note that our debt bound constraint

$$S(z^t,\eta^t) \leq Y(z^t)\underline{M},$$

is equivalent to

$$\widehat{S}(z^t, \eta^t) \leq \underline{M}$$

in our growth-deflated economy.

Multipliers: The consumer's marginal rate of substitution was given by

$$\beta \left(\frac{c(z^t, \eta^t)}{c(z^{t-1}, \eta^{t-1})} \right)^{-\alpha} \pi(z^t, \eta^t | z^{t-1}, \eta^{t-1}) = \frac{\zeta(z^t, \eta^t) P(z^t)}{\zeta(z^{t-1}, \eta^{t-1}) P(z^{t-1})} \pi(z^t, \eta^t | z^{t-1}, \eta^{t-1}).$$

Substituting in for the lhs, we see that , this becomes

$$\begin{aligned}
\beta \left(\frac{c(z^t, \eta^t)}{c(z^{t-1}, \eta^{t-1})} \right)^{-\alpha} \pi(z^t, \eta^t | z^{t-1}, \eta^{t-1}) &= \beta(z_{t-1}) \left(\frac{\widehat{c}(z^t, \eta^t) \exp(z_t)}{\widehat{c}(z^{t-1}, \eta^{t-1})} \right)^{-\alpha} \frac{\widehat{\pi}(z^t, \eta^t | z^{t-1}, \eta^{t-1})}{\exp\{z_t\}^{1-\alpha}} \\
&= \beta(z_{t-1}) \left(\frac{\widehat{c}(z^t, \eta^t)}{\widehat{c}(z^{t-1}, \eta^{t-1})} \right)^{-\alpha} \frac{\hat{\pi}(z^t, \eta^t | z^{t-1}, \eta^{t-1})}{\exp\{z_t\}}.
\end{aligned}$$

Similarly, substituting in for the rhs yields

$$\frac{\zeta(z^t, \eta^t) P(z^t)}{\zeta(z^{t-1}, \eta^{t-1}) P(z^{t-1})} \pi(z^t, \eta^t | z^{t-1}, \eta^{t-1}) = \frac{\zeta(z^t, \eta^t)}{\zeta(z^{t-1}, \eta^{t-1})} \frac{\hat{P}(z^t)}{\hat{P}(z^{t-1})} \frac{\hat{\pi}(z^t, \eta^t | z^{t-1}, \eta^{t-1})}{\exp\{z_t\}}.$$

Thus, our marginal rate condition in the original economy implies that

$$\beta(z_{t-1}) \left(\frac{\widehat{c}(z^t, \eta^t)}{\widehat{c}(z^{t-1}, \eta^{t-1})} \right)^{-\alpha} \hat{\pi}(z^t, \eta^t | z^{t-1}, \eta^{t-1}) = \frac{\widehat{\zeta}(z^t, \eta^t)}{\widehat{\zeta}(z^{t-1}, \eta^{t-1})} \frac{\widehat{P}(z^t)}{\widehat{P}(z^{t-1})} \hat{\pi}(z^t, \eta^t | z^{t-1}, \eta^{t-1}),$$

if

$$\frac{\zeta(z^t, \eta^t)}{\zeta(z^{t-1}, \eta^{t-1})} = \frac{\hat{\zeta}(z^t, \eta^t)}{\hat{\zeta}(z^{t-1}, \eta^{t-1})},$$

as we have so assumed. ■

To understand why the stationary analog is set up the way it is, note that we've deflated the quantity variables by the growth factor, adjust the discount factor $\hat{\beta}(z)$ to take account of the expected future growth on payoffs, and we've "twisted" the probabilities $\hat{\pi}(z)$ in a similar manner. This then leads to the following relationship between relative prices:

$$\begin{aligned}
\widehat{\pi}(z_{t+1}, \eta_{t+1} | z_t, \eta_t) \frac{\hat{P}(z^{t+1})}{\hat{P}(z^t)} &= \widehat{\pi}(z_{t+1}, \eta_{t+1} | z_t, \eta_t) \widehat{\beta}(z) \left(\frac{\hat{C}(z^{t+1})}{\hat{C}(z^t)} \right)^{-\alpha} \left(\frac{h(z^{t+1})}{h(z^t)} \right)^{\alpha} \\
&= \pi(z_{t+1}, \eta_{t+1} | z_t, \eta_t) \exp\{z_{t+1}\} \beta \left(\frac{C(z^{t+1})}{C(z^t)} \right)^{-\alpha} \left(\frac{h(z^{t+1})}{h(z^t)} \right)^{\alpha} \\
&= \pi(z_{t+1}, \eta_{t+1} | z_t, \eta_t) \exp\{z_{t+1}\} \frac{P(z^{t+1})}{P(z^t)}.
\end{aligned}$$

Also, note that since our first order conditions imply that in the non-stationary economy

$$\beta\left[\frac{c(z^{t+1},\eta^{t+1})}{c(z^t,\eta^t)}\right]^{-\alpha}\pi(z^{t+1},\eta^{t+1}|z^t,\eta^t)=\frac{\zeta(z^{t+1},\eta^{t+1})P(z^{t+1})}{\zeta(z^t,\eta^t)P(z^t)}\pi(z^{t+1},\eta^{t+1}|z^t,\eta^t),$$

this also implies that

$$\begin{aligned}&\widehat{\beta}(z_t)\exp(z_{t+1})^{1-\alpha}\beta\left[\frac{c(z^{t+1},\eta^{t+1})/Y(z^{t+1})}{c(z^t,\eta^t)/Y(z^t)}\right]^{-\alpha}\pi(\eta^{t+1}|z^{t+1},\eta^t)\widehat{\pi}(z_{t+1}|z_t)\\=\;&\widehat{\beta}(z_t)\exp(z_{t+1})\frac{\zeta(z^{t+1},\eta^t)P(z^{t+1})}{\zeta(z^t,\eta^t)P(z^t)}\pi(\eta^t|z^t,\eta^{t-1})\widehat{\pi}(z_t|z_{t-1}),\end{aligned}$$

where we have multiplied both sides by $\exp(z_{t+1})\widehat{\beta}(z_t)\pi(\eta^t|z^t,\eta^{t-1})\widehat{\pi}(z_t|z_{t-1})$ and divided by $\pi(z^{t+1},\eta^{t+1}|z^t,\eta^t)$. We can rewrite this expression as

$$\begin{aligned}&\widehat{\beta}(z_t)\exp(z_{t+1})^{1-\alpha}\left[\frac{\hat{c}(z^{t+1},\eta^{t+1})}{\hat{c}(z^t,\eta^t)}\right]^{-\alpha}\pi(\eta^{t+1}|z^{t+1},\eta^t)\widehat{\pi}(z_{t+1}|z_t)\\=\;&\frac{\hat{\zeta}(z^{t+1},\eta^t)\hat{P}(z^{t+1})}{\hat{\zeta}(z^t,\eta^t)\hat{P}(z^t)}\pi(\eta^t|z^t,\eta^{t-1})\widehat{\pi}(z_t|z_{t-1}),\end{aligned}$$

which is the equivalent expression in the stationary economy.

2 Stationary Optimality and Measurability Conditions

This section show how can we map the optimality and measurability conditions from nonstationary economy into stationary economy.

2.1 Complete Market Trader

2.1.1 Optimality condition

If borrowing constraints do not bind, the optimality condition is

$$\zeta(z^t,\eta^t)=\zeta(z^{t+1},\eta^{t+1})$$

By the first order condition respect to consumption, the optimality condition becomes

$$\begin{aligned}c(z^t,\eta^t)^{-\alpha}&=\frac{\zeta(z^t,\eta^t)}{h(z^t)^{-\alpha}}C(z^t,\eta^t)^{-\alpha}\\c(z^{t+1},\eta^{t+1})^{-\alpha}&=\frac{\zeta(z^{t+1},\eta^{t+1})}{h(z^{t+1})^{-\alpha}}C(z^{t+1},\eta^{t+1})^{-\alpha}\end{aligned}$$

In stationary economy,

$$\frac{c(z^t,\eta^t)}{Y(z^t)} = \widehat{c}(z^t,\eta^t)$$

$$\frac{\widehat{c}(z^t,\eta^t)^{-\alpha}}{Y(z^t)^{-\alpha}} Y(z^t)^{-\alpha} h(z^t)^{-\alpha} = \frac{\widehat{c}(z^{t+1},\eta^{t+1})^{-\alpha}}{Y(z^{t+1})^{-\alpha}} Y(z^{t+1})^{-\alpha} h(z^{t+1})^{-\alpha}$$

$$\widehat{c}(z^t,\eta^t) = \widehat{c}(z^{t+1},\eta^{t+1}) \frac{h(z^{t+1})}{h(z^t)} \tag{11}$$

2.1.2 Measurability condition

Note that there is no measurability condition for complete market trader.

2.2 Z-complete Trader

2.2.1 Optimality condition

If the borrowing constraints do not bind, the optimality condition for Z-complete trader becomes

$$\zeta(z^t,\eta^t) = \sum_{\eta^{t+1}} \zeta(z^{t+1},\eta^{t+1}) \pi(\eta^{t+1} \left| z^{t+1}, \eta^t \right.)$$

The stationary version of optimality condition is given by

$$\widehat{c}(z^t,\eta^t)^{-\alpha} = \sum_{\eta^{t+1}} \widehat{c}(z^{t+1},\eta^{t+1})^{-\alpha} \left(\frac{h(z^{t+1})}{h(z^t)} \right)^{-\alpha} \pi(\eta^{t+1} \left| z^{t+1}, \eta^t \right.) \tag{12}$$

2.2.2 Measurability Conditions

The measurability condition requires for all $\widetilde{\eta}^{t+1}$ and η^{t+1}

$$S(\zeta(z^{t+1},\eta^{t+1}); z^{t+1}, \eta^{t+1}) = S(\zeta(z^{t+1},\widetilde{\eta}^{t+1}); z^{t+1}, \widetilde{\eta}^{t+1})$$

So the stationary measurability condition is given by

$$\widehat{S}(\zeta(z^{t+1},\eta^{t+1}); z^{t+1}, \eta^{t+1}) = \widehat{S}(\zeta(z^{t+1},\widetilde{\eta}^{t+1}); z^{t+1}, \widetilde{\eta}^{t+1}) \text{ for all } \widetilde{\eta}^{t+1}, \eta^{t+1} \tag{13}$$

2.3 Diversified Trader

2.3.1 Optimality condition

If the borrowing constraints do not bind, the optimality condition for diversified trader becomes

$$\begin{aligned}&\sum_{z^{t+1},\eta^{t+1}} \zeta(z^{t+1},\eta^{t+1}) h(z^{t+1})^{\alpha} \exp(z_{t+1})^{-\alpha} \left[(1-\gamma)Y(z^{t+1}) + Q(z^{t+1})\right] \pi(z^{t+1},\eta^{t+1} \left| z^t,\eta^t \right.) \\ = \; & \zeta(z^t,\eta^t) \sum_{z^{t+1},\eta^{t+1}} \exp(z_{t+1})^{-\alpha} h(z^{t+1})^{\alpha} \left[(1-\gamma)Y(z^{t+1}) + Q(z^{t+1})\right] \pi(z^{t+1},\eta^{t+1} \left| z^t,\eta^t \right.)\end{aligned}$$

Rearranging the equation we obtain

$$\zeta(z^t,\eta^t) = \sum_{z^{t+1},\eta^{t+1}} \zeta(z^{t+1},\eta^{t+1}) \frac{\widehat{\pi}(z^{t+1},\eta^{t+1} \,|z^t,\eta^t)\left[(1-\gamma)Y(z^{t+1}) + Q(z^{t+1})\right]\left(\frac{h(z^{t+1})}{h(z^t)}\right)^{\alpha}}{\sum_{z^{t+1}} \widehat{\pi}(z^{t+1} \,|z^t)\left[(1-\gamma)Y(z^{t+1}) + Q(z^{t+1})\right]\left(\frac{h(z^{t+1})}{h(z^t)}\right)^{\alpha}}$$

The stationary measurability conditions are given by

$$\widehat{c}(z^t,\eta^t)^{-\alpha} = \sum_{z^{t+1},\eta^{t+1}} \widehat{c}(z^{t+1},\eta^{t+1})^{-\alpha} \left(\frac{h(z^{t+1})}{h(z^t)}\right)^{-\alpha} \frac{\widehat{\pi}(z^{t+1},\eta^{t+1} \,|z^t,\eta^t)\left[(1-\gamma) + \widehat{Q}(z^{t+1})\right]\left(\frac{h(z^{t+1})}{h(z^t)}\right)^{\alpha}}{\sum_{z^{t+1}} \widehat{\pi}(z^{t+1} \,|z^t)\left[(1-\gamma) + \widehat{Q}(z^{t+1})\right]\left(\frac{h(z^{t+1})}{h(z^t)}\right)^{\alpha}} \tag{14}$$

2.3.2 Measurability Condition

The measurability condition requires for all $\widetilde{z}^{t+1},\widetilde{\eta}^{t+1}$ and z^{t+1},η^{t+1}

$$\frac{S(z^{t+1},\eta^{t+1})}{\left[(1-\gamma)Y(z^{t+1}) + Q(z^{t+1})\right]} = \frac{S(\widetilde{z}^{t+1},\widetilde{\eta}^{t+1})}{\left[(1-\gamma)Y(\widetilde{z}^{t+1}) + Q(\widetilde{z}^{t+1})\right]}$$

The stationary measurability conditions are given by

$$\frac{\widehat{S}(z^{t+1},\eta^{t+1})}{\widehat{Q}_{t+1}(z^{t+1}) + (1-\gamma)} = \frac{\widehat{S}(\widetilde{z}^{t+1},\widetilde{\eta}^{t+1})}{\widehat{Q}_{t+1}(\widetilde{z}^{t+1}) + (1-\gamma)} \tag{15}$$

for all $\widetilde{z}^{t+1},\widetilde{\eta}^{t+1}$ and z^{t+1},η^{t+1}

2.4 Non-participant

2.4.1 Optimality Conditions

If borrowing constraints do not bind, the optimality condition for non-participant becomes

$$\zeta(z^t,\eta^t) = \sum_{z^{t+1},\eta^{t+1}} \zeta(z^{t+1},\eta^{t+1}) \frac{\pi(z^{t+1},\eta^{t+1}\,|z^t,\eta^t\,)P(z^{t+1})}{\sum_{z^{t+1}} \pi(z^{t+1}\,|z^t\,)P(z^{t+1})}$$

Rearranging the equation

$$\zeta(z^t,\eta^t) = \sum_{z^{t+1},\eta^{t+1}} \zeta(z^{t+1},\eta^{t+1}) \frac{\exp(z_{t+1})^{-1}\widehat{\pi}(z^{t+1},\eta^{t+1}\,|z^t,\eta^t\,)\left(\frac{h(z^{t+1})}{h(z^t)}\right)^{\alpha}}{\sum_{z^{t+1}} \exp(z_{t+1})^{-1}\widehat{\pi}(z^{t+1}\,|z^t\,)\left(\frac{h(z^{t+1})}{h(z^t)}\right)^{\alpha}}$$

Which gives the stationary measurability condition

$$\widehat{c}(z^t,\eta^t)^{-\alpha} = \sum_{z^{t+1},\eta^{t+1}} \widehat{c}(z^{t+1},\eta^{t+1})^{-\alpha}\left(\frac{h(z^{t+1})}{h(z^t)}\right)^{-\alpha} \frac{\exp(z_{t+1})^{-1}\widehat{\pi}(z^{t+1},\eta^{t+1}\,|z^t,\eta^t\,)\left(\frac{h(z^{t+1})}{h(z^t)}\right)^{\alpha}}{\sum_{z^{t+1}} \exp(z_{t+1})^{-1}\widehat{\pi}(z^{t+1}\,|z^t\,)\left(\frac{h(z^{t+1})}{h(z^t)}\right)^{\alpha}} \tag{16}$$

2.4.2 Measurability Conditions

The measurability condition requires for all $\widetilde{z}^{t+1},\widetilde{\eta}^{t+1}$ and z^{t+1},η^{t+1}

$$S(\zeta(z^{t+1},\eta^{t+1});z^{t+1},\eta^{t+1}) = S(\zeta(\widetilde{z}^{t+1},\widetilde{\eta}^{t+1});\widetilde{z}^{t+1},\widetilde{\eta}^{t+1})$$

Hence, the stationary measurability condition is for all $\widetilde{z}^{t+1},\widetilde{\eta}^{t+1}$ and z^{t+1},η^{t+1}

$$\frac{\widehat{S}(\zeta(z^{t+1},\eta^{t+1});z^{t+1},\eta^{t+1})}{\exp(z^{t+1})} = \frac{\widehat{S}(\zeta(\widetilde{z}^{t+1},\widetilde{\eta}^{t+1});\widetilde{z}^{t+1},\widetilde{\eta}^{t+1})}{\exp(\widetilde{z}^{t+1})} \tag{17}$$

3 Computational Algorithm

We use a finite history of length n of the aggregate shocks to (reasonably) accurately compute the equilibrium. The variable n determines the set of aggregate finite histories $Z(n)$ that we are keeping track of, and $z^n \in Z(n)$ denotes a specific n-period history event. The number of elements of $S(n)$ is given by $(\#z)^n$, where $\#z$ is the number of aggregate states. The individual state is then given by his multiplier, the finite aggregate history, and his individual shock; besides his multiplier, there are $(\#z)^n \times \#\eta$ states for the individual, where $\#\eta$ is the number of idiosyncratic states

The algorithm works as follows. Assume that we have a matrix $H = h(z')/h(z^n)$ with dimension $(\#z)^n$ by $\#z$, which gives the value of our moment $h(z')/h(z^n)$ in the case where the transition is from finite history z^n to finite history (z^n, z'). Given this matrix we can compute the aggregate state price in the stationary version of the economy, which we will denote by $\hat{P}(z')/\hat{P}(z^n)$. In computing the equilibrium, we find it more convenient to keep track of agents by their consumption share $\widehat{c}(z^n, \eta)$ rather than their (normalized) multiplier $\zeta(z^n, \eta)$. Note that $\widehat{c}(z^n, \eta)^{-\alpha} h(z^n)^{-\alpha} = \zeta(z^n, \eta)$.

We start the computation by guessing the saving function $\widehat{S}(\widehat{c}, z, \eta)$ for each type of trader, called $\widehat{S}_j(\widehat{c}, z, \eta)$ where j denotes for number of iterations. Then, to compute $\widehat{S}_{j+1}$ given $\widehat{S}_j$ we use the following algorithm:

1. We start with a savings grid, where the highest savings level is the debt/savings limit. Note that since this is a fraction of the net present value of income, we can compute this directly given $h(z^{t+1})/h(z^t)$.

2. For each type of trader and given each savings grid point sg_i, we can compute the associated consumption share in next period $\widehat{c}'(s', \eta')$ by $sg_i = \widehat{S}_j(\widehat{c}'(s', \eta'), s', \eta')$. Note that we need to impose measurability conditions on the saving function which depend on the type of trader.

3. Given sg_i and it's corresponding consumption share $\widehat{c}'(s', \eta')$, we can compute the consumption share today from the optimality condition for today's state (s, η). Take Z-complete trader as an example, this is given by

$$\widehat{c}^{-\alpha} = \sum_{\eta^{t+1}} \widehat{c}'(z', \eta')^{-\alpha} \left(\frac{h(z')}{h(z)} \right)^{-\alpha} \pi(\eta' \,|z', \eta)$$

   If we do this for every point in tomorrow's savings grid, fixing the state today (z, η), this yields a vector of current consumption shares $\mathbf{c}$ and their future associated net savings levels $\mathbf{S}'$ for each possible transition (z', η'). Hence we have constructed a mapping between current consumption shares and next period net saving levels $\widehat{D}'(c; z', \eta')$.

4. Given $\widehat{D}'(\widehat{c}; z', \eta')$, we can form an updated guess for the saving function

$$\widehat{S}_{j+1}(\widehat{c}; z^n, \eta) = [\gamma\eta - \widehat{c}] + \sum_{z', \eta'} \widehat{\pi}(z', \eta' \,|z^n, \eta) \widehat{\beta}(z^n) \left(\frac{h(z')}{h(z^n)} \right)^{\alpha} \widehat{D}'(\widehat{c}; z', \eta')$$

. In so doing we have constructed the updated saving function $\widehat{S}_{j+1}(\widehat{c}; z^n, \eta)$.

5. The iterations continues until the $\widehat{S}_j(\widehat{c}; z^n, \eta)$ functions converge. As one of the products of this computation we have the vectors $\mathbf{c}$ and $\mathbf{c}'(z', \eta')$ for each transition (z', η'). We store these vectors in an array and use them in our simulation step when we update the values of H implied by our transition functions for consumption shares.

6. To simulate our economy and update H, we take a single panel draw of aggregate and idiosyncratic shocks. We then compute the updated consumption shares, where each period we normalize the consumption shares to average 1, and use the normalization factor to generate a revised estimate H'. If H' is sufficiently close to H, the algorithm terminates, otherwise we start again using H' as the new starting guess for H.

4 The program codes

This section shows some details of the program code and explains how to implement those optimality and measurability conditions in previous section. There are two main m-files to solve the model. They are `environment_segmented.m` and `H_update.m`. (They are available in author's webpage)

4.1 environment_segmented.m

This file sets the parameters that will be used by the other m-files. Most importantly, the stochastic processes are defined here. We form an aggregate history transition matrix and probabilities of the idiosyncratic shock conditional on the aggregate state. These are used to construct a time series of aggregate shocks and a panel of idiosyncratic shocks consistent with the aggregate series.

4.1.1 Histories and Transitions

Having set the transition matrix between each state (η, z) as `Trans_total` with associated invariant distribution `Inv_total,` we aggregate across idiosyncratic states η to form the aggregate transition matrix `Ztrans` which is square with dimension `Znum`, the number of aggregate states.

We specify `Lagnum` which is the length of the history that agents remember. `Ztotal=Znum^Lagnum` is the dimension of the aggregate state space and `Znum_total = Ztotal*Enum` is the dimension of the state for each agent, since for each agent the state variable is the truncated aggregate history and the agent's current idiosyncratic shock $(\{z\}, \eta)$. `Zlag` is the matrix of all possible aggregate histories.

`B_hat` is the growth adjusted discounted factor.

$$\texttt{B\_hat}\,(z'|z) \quad = \beta \sum_{z'} \pi(z'|z)(\exp\{z'\})^{1-\alpha}$$

4.1.2 Grids and Twisted Probabilities

Two linear grids, `dimgrid2` and `dimgrid3` are constructed. They are consumption share grid point corresponding to savings function and inverse savings function.

A random panel is constructed with `N-1` agents and `T` time periods. We form the twisted probabilities using the equation

$$\texttt{Z\_hat}\,(z'|z) = \frac{\pi\,(z'|z)\, z'^{1-\gamma}}{\sum_{z'} \pi\,(z'|z)\, z'^{1-\gamma}}$$

`Z_hat_total` is the same idea but with a probability for each transition $(\eta, z) \to (\eta', z')$

$$\texttt{Z\_hat\_total}\,(z', \eta'|z, \eta) = \frac{\pi\,(z', \eta'|z, \eta)\, z'^{1-\gamma}}{\sum_{z'.\eta'} \pi\,(z', \eta'|z, \eta)\, z'^{1-\gamma}}$$

Given an aggregate history $\{z\}_t \equiv (z_{t-\text{Lagnum}+1}, ..., z_{t-1}, z_t)$ there are only `Znum` possibilities for the aggregate history next period. The transition matrix from one aggregate history to the next will thus be `Ztotal`×`Ztotal` but with each row having at most `Znum` nonzero entries. `P_hat_indc` has ones in every spot where the aggregate history transition matrix has nonzero entries and `P_hat_final` is the transition matrix formed by replacing the ones with the appropriate probabilities from the aggregate transition matrix. Finally, `CE_hat` is the vector of conditional probabilities $\Pr(\eta'|z', \eta)$.

4.1.3 Panel Construction

`Z_state` 1×`T` stores the aggregate state z, `T_state` 1×`T` stores the truncated aggregate history $\{z\}$, `E_state` (`N`−1)×`T` stores the idiosyncratic state. The initial aggregate state is arbitrarily set to the first enumerated aggregate state, the initial aggregate history is arbitrarily set to the first enumerated initial history and the initial idiosyncratic state is drawn from the invariant distribution over idiosyncratic states conditional on being in the first aggregate state.

ZIND is a Ztotal$\times$2 matrix the i^{th} row of which gives the column indices of the nonzero entries of the i^{th} row of P_hat_indc. That is, given an aggregate history i, ZIND gives the two possible continuation histories next period. ZEIDC is similar in that for every state $(\{z\}, \eta)$ it gives the possible states $\left(\{z\}', \eta'\right)$.

For each point in time t, we randomly draw an aggregate shock z_{t+1}. This induces a new aggregate history $\{z\}_{t+1}$ which we determine using ZIND. Using CE_hat we randomly draw η_{t+1} from its distribution conditional on (z', η). The invariant distribution over idiosyncratic states puts half the agents in each of the two idiosyncratic states. We randomly reassign agents to new states so that exactly half the agents have each idiosyncratic shock.

4.2 H_update.m

We start with a guess for H, the aggregate multiplier updating rule in each aggregate state transition, and for each type a guess for the savings level at each level of consumption on the grid. Using linear interpolation this implies a piecewise linear function mapping contemporaneous consumption to savings, or inversely, mapping savings to consumption.

We construct a grid over future savings s'. For each type in each state, we use the inverse mapping to get consumption tomorrow c', and then the aggregate multiplier rule together with current period consumption give consumption today c. Finally, using c and s' we compute current period savings, s, as the the difference between current period income and consumption plus expected future savings. The new (c, s) pairs allow us to use piecewise linear interpolation to update the savings function for each type. We iterate this savings function updating until convergence for each type.

Using the savings function from the previous step, we can easily calculated consumption tomorrow as a function of consumption today for each type in each state transition.

Finally, we feed the panel of shocks constructed in environment_segmented_sty.m into the consumption updating rules to update the consumption share for each agent in each period. This implies a value for the aggregate multiplier H since we adjust H to maintain stationarity of consumption shares. We use the simulated values of H to form a new guess for H in each state transition. We continue iterating these steps until H converges at every point in the state space.

4.2.1 Setting Things Up

`MM` is a matrix with columns representing the fraction of each type of agent in the economy. We will strip off one column to get the fractions, which are labeled `omega`i $i \in \{1, 2, 3, 4\}$. $H\left(\{z\}_t, z_{t+1}\right)$ is the multiplier updating rule given that we are in aggregate history $\{z\}_t$ and we receive new aggregate shock z_{t+1}. In the code, this is represented by the `Ztotal`×`Znum` matrix `H` which is initialized with 1.01 in every entry. `H_total` is a `Ztotal`×`Ztotal` matrix with the entries from `H` raised to the power γ and moved to the spots corresponding to the nonzero entries in `ZIND`.

We form the `Enum*Ztotal`×`Enum*Ztotal` matrix `R_hat_total` to tabulate

$$\hat{R}\left(\left(\{z\}', \eta'\right), \left(\{z\}, \eta\right)\right) = \hat{\beta}\left(z\right)\hat{\pi}\left(z', \eta' | z, \eta\right)\texttt{H\_total}\left(\{z\}, \{z\}'\right)$$

where z is the last element in $\{z\}$ and z' the last element in $\{z\}'$. Note that this matrix is nonzero only at entries where $\{z\}' \succ \{z\}$. Similarly we form a `Ztotal`×2 matrix

$$\hat{R}_z\left(\{z\}, z'\right) = \beta\pi\left(z'|z\right)\left(z'\right)^{-\gamma} H\left(\{z\}, z'\right)$$

The risk free rate is given by

$$R^{free}\left(\{z\}\right) = \sum_{z'} \frac{1}{\hat{R}_z\left(\{z\}, z'\right)}$$

The price-dividend ratio for aggregate consumption is calculated as follows. First define

$$\texttt{ac\_temp}\left(\{z\}\right) = \sum_{z'} \hat{R}_z\left(\{z\}, z'\right)\left(1-\gamma\right) z'$$

which is the expected endowment of diversifiable income next period under the appropriate twisted probability measure. Then we solve the recursive pricing equation for

$$\texttt{ac\_pd}\left(\{z\}\right) = \left(I - A\right)^{-1}\texttt{ac\_temp}\left(\{z\}\right)$$

where A is a `Ztotal`×`Ztotal` matrix with

$$A\left(\{z\}, \{z\}'\right) = \begin{cases} z'\hat{R}_z\left(\{z\}, z'\right) & \text{if } \{z\}' \succ \{z\} \\ 0 & \text{otherwise} \end{cases}$$

Next we form the twisted probabilities for each type of agent. First define

$$\hat{R}_{\text{tmp2}}\left(\eta', z' | \{z\}, \eta\right) = \hat{R}\left(\left(\{z\}', \eta'\right), \left(\{z\}, \eta\right)\right) * \left[ac\_pd\left(\{z\}\right) + \left(1-\gamma\right)\right]$$

The twisted probabilities for the capital trader are

$$\hat{\pi}_{\text{cap}}\left(\eta', \{z\}' \mid \{z\}, \eta\right) = \frac{\hat{R}_{\text{tmp2}}\left(\eta', \{z\}' \mid \{z\}, \eta\right)}{\sum_{\{z\}} \hat{R}_{\text{tmp2}}\left(\eta', \{z\}' \mid \{z\}, \eta\right)}$$

and the twisted probability for the bond trader are

$$\hat{\pi}_{\text{bond}} = \frac{\hat{R}\left(\left(\{z\}', \eta'\right), \left(\{z\}, \eta\right)\right)/z'}{\sum_{(z',\eta')} \hat{R}\left(\left(\{z\}', \eta'\right), \left(\{z\}, \eta\right)\right)/z'}$$

where we maintain our convention that z' is the last element in $\{z\}'$.

Using these twisted probabilities between states we can calculate the present value of income.

$$\texttt{NPV\_inc} = \left(I - \hat{R}\right)^{-1} \texttt{Income}$$

We then have the borrowing limit in every period

$$\texttt{Dbound} \;=\; M * \texttt{NPV\_inc}$$

Now we form the initial annuity value of savings. For each agent, in each state $(\{z\}, \eta)$ at each point in the consumption grid we set $S(\{z\}, \eta) = \left(I - \hat{R}\right)^{-1} (\texttt{Income} - \texttt{Consumption})$. The matrices storing these values are calls Sav_com, Sav_z, Sav_cap, and Sav_bond for the different types of traders. Now we are ready to update these savings functions.

4.2.2 Updating the savings function

Complete Asset Trader Define Dbound_low_com to be the minimum value across the grid in each state $(\{z\}, \eta)$.

We loop over every state $(\{z\}, \eta)$. We then loop over every state tomorrow (z', η'). H next period is found by setting H_next= $H(\{z\}, z')$. The enumeration of tomorrow's sate $\left(\{z\}', \eta'\right)$ is given by plugging in $(\{z\}, \eta), (z', \eta')$ to the ZEIDC matrix. We use this enumeration to set the lower bound Dbound_low_tmp = Dbound_low_com$\left(\{z\}', \eta'\right)$ and to get upper bound Dbound$\left(\{z\}', \eta'\right)$ and construct a linear grid of savings between these values. In each state $\left(\{z\}', \eta'\right)$ we have savings tomorrow as a function of consumption tomorrow, and now we invert this relationship to get consumption as a function of savings which we denote $c'_{\left\{\{z\}', \eta'\right\}}(s')$. Using this, for each level of savings tomorrow, s', in the grid we compute the associated level of consumption this period, c_tmp, by

$$\texttt{c\_tmp} = \texttt{H\_next} \times C'_{\left(\{z\}', \eta'\right)}(s')$$

We throw out the cases where consumption goes negative. We have now constructed a piecewise linear function of savings tomorrow as a function of consumption today, call it $S'_{(\{z\}',\eta')}(c)$. Tomorrow's savings are constructed as a function of consumption (for each c in the consumption grid) by setting

$$\texttt{Spr}\left(\left(\{z\},\eta\right),\left(z',\eta'\right),c\right)=\begin{cases} S'_{(\{z\}',\eta')}(c) & \text{if } c>\underline{c} \\ \texttt{Dbound} & \text{otherwise}\end{cases}$$

where $\underline{c}$ is the level of consumption when savings is at Dbound. That is, $c \leq \underline{c}$ would imply exceeding the boundary of the grid.

Having constructed $\texttt{Spr}\left(\left(\{z\},\eta\right),\left(z',\eta'\right),c\right)$ we can constructed savings in each state this period as

$$\texttt{Sav\_com\_new}\left(c,\left(\{z\},\eta\right)\right)=\texttt{Income}-c+\sum_{\eta',z'}\texttt{Spr}\left(\left(\{z\},\eta\right),\left(z',\eta'\right),c\right)\hat{R}_{\text{tmp2}}\left(\eta',\{z\}'\,|\,\left(\{z\},\eta\right)\right)$$

Thus we have constructed a new set of points (c,s) that when interpolated form a (updated) piecewise linear savings function.

We store the maximum absolute difference between the old and new savings function, disregarding the last 5% of entries, and then set Sav_com equal to Sav_com_new. If the difference is large, we repeat this step, otherwise we move on to the next type.

Z-complete Trader This is the same as above with the following consumption computed as follows.

$$\texttt{c\_tmp}=\texttt{H\_next}\times\left[\sum_{\eta'}\Pr\left(\eta'|z',\eta\right)\left(c_{(\{z\}',\eta')}(s)\right)^{-\gamma}\right]^{-1/\gamma}$$

Capital Trader This is the same as above with the following differences. We start by looping over every state today $(\{z\},\eta)$. We define

$$\texttt{Dbound\_tmp}\left(z',\eta'\right)=\left[\texttt{ac\_pd\_all}(\{z\}',\eta')+(1-\gamma)\right]\min_{(\{\tilde{z}\}',\tilde{\eta}')\succ(\{z\},\eta)}\frac{\texttt{Dbound}\left(\{\tilde{z}\}',\tilde{\eta}'\right)}{\texttt{ac\_pd\_all}(\{\tilde{z}\}',\tilde{\eta}')+(1-\gamma)}$$

and

$$\texttt{Wbound=ac\_pd\_all}(\{z\},\eta)\min_{(\{\tilde{z}\}',\tilde{\eta}')\succ(\{z\},\eta)}\frac{\texttt{Dbound}\left(\{\tilde{z}\}',\tilde{\eta}'\right)}{\texttt{ac\_pd\_all}(\{\tilde{z}\}',\tilde{\eta}')+(1-\gamma)}$$

We construct a linear grid of savings next period from `Wtmp` to `min(Dbound_low_cap)`. For each grid point s' in `Wtmp` we calculate the associated level of adjusted savings $\tilde{s}'$.

$$\texttt{Spr\_tmp}\left(z',\eta'\right) = \frac{\tilde{s}'}{\texttt{ac\_pd\_all(}\left\{z\right\},\eta\texttt{)}} \texttt{*[ac\_pd\_all(}\left\{z\right\}',\eta'\texttt{)+}\left(1-\gamma\right)\texttt{]}$$

That is, the grid of values of s' induces a grid of values of tomorrow's adjusted savings $\tilde{s}'$. Let $C'_{\left(\{z\}',\eta'\right)}\left(\tilde{s}'\right)$ be the inverse savings function where the savings function is defined by linearly interpolating pairs of adjusted savings levels stored in `Sav_cap`(z',η') and consumption levels in `mgrid2`. Now we calculate tomorrow's consumption c' for each point in the grid and for each possible state tomorrow as

$$\texttt{cpr\_tmp}\left(\tilde{s}',\left(z',\eta'\right)\right) = C'_{\left(\{z\}',\eta'\right)}\left(\tilde{s}'\right)$$

and use the updating rule to calculate current period consumption c and noting that we have a one to one mapping between s' and $\tilde{s}'$ we calculate

$$\texttt{c\_tmp}\left(s'\right) = \left(\sum_{\left(z',\eta'\right)} H\left(\left\{z\right\},z'\right)\left(\texttt{cpr\_tmp}\left(\tilde{s}',\left(z',\eta'\right)\right)\right)^{-\gamma}\hat{\pi}_{\text{cap}}\left(\eta',z'\middle|\left\{z\right\},\eta\right)\right)^{-1/\gamma}$$

We have now constructed a piecewise linear function of savings tomorrow as a function of consumption today, call it $S'\left(c\right)$ and note that this does not depend on next period's shock. Define for each consumption grid point c, the promised savings level

$$\texttt{Spr}\left(\left(\left\{z\right\},\eta\right),c\right) = \begin{cases} S'\left(c\right) & \text{if } c > \underline{c} \\ \chi\left(\left\{z\right\},\eta\right) & \text{otherwise} \end{cases}$$

where

$$\chi\left(\left\{z\right\},\eta\right) = s_{\max}\sum_{\left(z',\eta'\right)}\hat{R}\left(\left(z',\eta'\right)\middle|\left(\left\{z\right\},\eta\right)\right)\frac{\texttt{ac\_pd\_all(}\left\{z\right\}',\eta'\texttt{)+}\left(1-\gamma\right)}{\texttt{ac\_pd\_all(}\left\{z\right\},\eta\texttt{)}}$$

Finally we have

$$\texttt{Sav\_cap\_new}\left(\left(\left\{z\right\},\eta\right),c\right) = \texttt{Income} - c + \texttt{Spr}\left(\left(\left\{z\right\},\eta\right),c\right)$$

Nonparticipant This is the same as above with the following differences. Define

$$\texttt{Dbound\_tmp} = \min_{z',\eta'}\texttt{Dbound}\left(\left\{z'\right\},\eta'\right)\frac{z'}{z_{\min}}$$

where $z_{\min}$ is the minimum value of the aggregate shock and let W_tmp be the grid from Dbound_tmp to the minimum value of Dbound_tmp_bound. For each grid point s' in W_tmp and for each state next period we have adjusted savings $\tilde{s}'$

$$\texttt{Spr\_tmp}\left(\left(z',\eta'\right),s'\right)=\frac{s'}{z'/z_{\min}}$$

and c' =cpr_tmp$(s',(z',\eta'))$ is formed as above by evaluating the inverse savings function at each adjusted savings level $\tilde{s}'$. We calculate current period consumption, c, as

$$\texttt{c\_tmp}\left(s'\right)=\left(\sum_{\eta',z'}(H\left(\left\{z\right\},z'\right)*\texttt{cpr\_tmp}\left(s',\left(z',\eta'\right)\right))^{-\gamma}\hat{\pi}_{\text{bond}}\left(\left(z',\eta'\right)\mid\left(\left\{z\right\},\eta\right)\right)\right)^{-1/\gamma}$$

Next we calculate promised savings

$$\texttt{Spr}\left(\left(\left\{z\right\},\eta\right),c'\right)=\begin{cases}S'\left(c\right) & \text{if } c>\underline{c}\\ \chi\left(\left\{z\right\},\eta\right) & \text{otherwise}\end{cases}$$

where

$$\chi\left(\left\{z\right\},\eta\right)=s_{\max}\sum_{(z',\eta')}\hat{R}\left(\left(z',\eta'\right)\mid\left(\left\{z\right\},\eta\right)\right)\frac{z'}{z_{\min}}$$

Finally, the new savings function is given by

$$\texttt{Sav\_bond\_new}\left(\left(\left\{z\right\},\eta\right),c\right)=\texttt{Income}-c+\texttt{Spr}\left(\left(\left\{z\right\},\eta\right),c\right)$$

4.3 Calculating Consumption Functions

The idea here is to use the savings functions determined in the last section to create an approximate function that gives c' as a function of c for each type. We will use these functions in the next subsection where we simulate to update H.

Complete Trader We loop over the state today $(\{z\},\eta)$ and the possible shocks tomorrow (z',η'). We form cpr_tmp which is c', and c_tmp which is c for each point in the savings grid, exactly as we did for the complete trader above. Then we form consumption next period c' at each grid point c_{grid} in mgrid2

$$\texttt{Cpr\_com}_{(\{z\},\eta)}\left(c_{grid},\left(z',\eta'\right)\right)=\begin{cases}g\left(c_{grid}\right) & \text{if } c_{grid}>\underline{c}\\ \underline{c} & \text{otherwise}\end{cases}$$

where g is the function formed by linearly interpolating between the the (c_tmp, cpr_tmp) pairs.

z-complete Trader We loop over the state today $(\{z\}, \eta)$ and the possible aggregate states tomorrow z'. We begin as before, but for each of the two possible idiosyncratic shocks, call them η_1 and η_2, we construct consumption from the inverse savings function with that shock. That is for each point s in the savings grid we set

$$\texttt{cpr1=}C'_{\{z\}',\eta'_1}(s) \qquad \text{and} \qquad \texttt{cpr2=}C'_{\{z\}',\eta'_2}(s)$$

[Note that this will only work when `Enum`= 2]. Then we have

$$\texttt{c\_tmp=}\left(\Pr(\eta' = \eta_1|z', \eta)\,\texttt{cpr1}^{-\gamma}\texttt{+}\Pr(\eta' = \eta_1|z', \eta)\,\texttt{cpr2}^{-\gamma}\right)^{-1/\gamma}$$

We compute c' at each grid point by

$$\texttt{Cpr\_z}_{(\{z\},\eta)}(c_{grid}, (z', \eta')) = \begin{cases} g(c_{grid}) & \text{if } c_{grid} > \underline{c} \\ \underline{c} & \text{otherwise} \end{cases}$$

where g is constructed by using `c_tmp` and `cpr1` if $\eta' = \eta_1$ and by using `c_tmp` and `cpr2` if $\eta' = \eta_2$.

Capital Trader We follow exactly the steps described in the previous capital trader subsection. Then we form

$$\texttt{Cpr\_cap}_{(\{z\},\eta)}(c_{grid}, (z', \eta')) = \begin{cases} g(c_{grid}) & \text{if } c_{grid} > \underline{c} \\ \underline{c} & \text{otherwise} \end{cases}$$

where g is constructed using `c_tmp` and `cpr_tmp`.

Bond Trader We follow exactly the steps described in the previous bond trader subsection. Then we form $\texttt{Cpr\_bond}_{(\{z\},\eta)}(c_{grid}, (z', \eta'))$ analogously to how it is formed for the capital trader.

4.3.1 Updating H

Here we simulate the economy to update the function H that describes the law of motion for the aggregate multiplier.

The Simulation There are $N-1$ agents and T time periods. The variable `C_STATE` is $(N-1)\times T$ and stores the consumption share for each agent in each time period. `H_STATE` stores the updated aggregate multiplier. To get the right fraction of each type of agent we define

$$N_1 = (N-1) * \texttt{omega1}$$
$$N_2 = (N-1) * (\texttt{omega1+omega2})$$
$$N_3 = (N-1) * (\texttt{omega1+omega2+omega3})$$

We loop over time t. `pick` is the $N \times \texttt{Enum}^2$ matrix where each row has a one in the column corresponding to the enumeration of (η_t, η_{t+1}) and zeros elsewhere. For each agent type j we compute for each state

$$\texttt{cp\_tmp}\,(w, (\eta, \eta')) = \texttt{Z\_state}_{t+1} * \texttt{Cpr\_j}_{(\{z\},\eta)}\,(w, (z', \eta'))$$

this gives consumption share tomorrow w' for each value of consumption share today w given each pair of shocks (η, η'). Then, we form a function $G_{\{\eta,\eta'\}} : w \mapsto w'$ by linearly interpolating between pairs (w, w'). We use this function to map the consumption share last period, w_t^i stored in `C_state`, into the consumption share today $w_{t+1}^i = G(w_t^i)$ for each individual which we call `cpr`. We set `cpr=min(cpr,Mmax)` and form

```
h_update = mean(cpr)
C_state(:,t+1) = cpr/h_update
H_STATE(1,t+1) = h_update*H(T_STATE(1,t),Z_STATE(1,t+1))
```

where the last expression uses the old policy rule for updating H to find the next value of H_{t+1} in the simulation.

Updating H We drop the first `TT` periods. Currently we have `TT=500` in the code. We loop over each truncated aggregate history $\{z\}$ and value of the aggregate shock next period, z'. We find `index` which is the position where $\{z\}, z'$ occurs in T_STATE and Z_STATE respectively. We define `H_new(i,j)=mean(H_STATE(index))` which is the mean of H over all realizations of the state $\{z\}, z'$ indexed by `(i,j)` in our enumeration of the states. If the maximum elementwise distance between `H` and `H_new` is large enough we set `H=`λ`H_new+`$(1-\lambda)$`H` and reupdate the savings, consumption and H matrices. When the distance is small enough, we are finished.