Lecture 2, Part B: Contingent Claims Analysis and Stochastic Discount Factors

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Roadmap

Contingent Claims

- Suppose that there are S states of nature tomorrow.
- Let s denote an individual state.

Definition

A contingent claim is a security that pays one dollar (or one unit of the consumption good) in one state s only tomorrow.

- The price today of this contingent claim is $P_c(s)$.
- In complete markets, investors can buy any contingent claim (or synthesize all contingent claims).

Pricing

- Let $X(s)$ denote an asset's payoff in state of nature s.
- Then the price of this asset is:

$$P(X) = \sum_{s=1}^{S} P_c(s) X(s).$$

- Let $\pi(s)$ be the probability that state s occurs:

$$P(X) = \sum_{s=1}^{S} \pi(s) \frac{P_c(s)}{\pi(s)} X(s).$$

- Then define M as the ratio of contingent claim price to probability:

$$M(s) = \frac{P_c(s)}{\pi(s)}.$$

Pricing

- Now, we are back to our familiar pricing equation:

$$P(X) = \sum_{s=1}^{S} \pi(s) M(s) X(s) = E(MX).$$

Risk-neutral probabilities

- Let us define $\pi(s)^\star$ as:

$$\pi(s)^\star \equiv R^f \pi(s) M(s) = R^f P_c(s),$$

where the gross risk free-rate R^f is:

$$R^f \equiv \frac{1}{\sum_{s=1}^{S} P_c(s)} = \frac{1}{E(M)}.$$

$\pi(s)^\star$ are probabilities:

1. they are positive
2. less than or equal to one
3. sum to one.

Risk-neutral probabilities

- We can rewrite the asset pricing equation as:

$$P(X) = \sum_{s=1}^{S} P_c(s) X(s) = \frac{1}{R^f} \sum_{s=1}^{S} \pi^\star(s) X(s) = \frac{E^\star(X)}{R^f},$$

where $E^\star$ denotes expectation under the risk-neutral probabilities $\pi^\star$ instead of real probabilities π.

- $\pi^\star$ give greater weight to states with higher than average marginal utility growth M.

Risk sharing

- we consider a two-period economy.
- the investor starts with initial wealth Y and a state-contingent income $Y(s)$.
- investors trade a complete menu of state-contingent claims

Risk sharing

- The investor's maximization problem:

$$Max_{\{C,C(s)\}}\, u(C) + \sum_{s=1}^{S} \beta\pi(s)u(C(s))$$

subject to:

$$C + \sum_{s=1}^{S} P_c(s)C(s) = Y + \sum_{s=1}^{S} P_c(s)Y(s).$$

- Let λ be the Lagrange multiplier on the budget constraint:

$$\begin{aligned} u'(C) &= \lambda \\ \beta\pi(s)u'(C(s)) &= \lambda P_c(s). \end{aligned}$$

Only Aggregate Risk Matters

- Eliminating λ:

$$
\begin{aligned}
P_c(s) &= \beta \pi(s) \frac{u'(C(s))}{u'(C)}, \text{ or,} \\
M(s) &= \frac{P_c(s)}{\pi(s)} = \beta \frac{u'(C(s))}{u'(C)}.
\end{aligned}
$$

- M is the marginal rate of substitution between date- and state-contingent commodities.
  - all investors equalize their IMRS to the contingent price ratio
  - all investors face same prices
- marginal utility growth is equalized across investors
- if they share the same preferences, then they equalize consumption growth as well

Roadmap

Existence of SDF

- suppose we observe a set of prices P and payoffs X
- markets are *incomplete*.
- Under what conditions can we represent prices as follows?:

$$P = E(MX)$$

Law of one price and existence of discount factor

- Let $\underline{X}$ be the payoff space.
- *Assumptions*:
  1. **Portfolio formation**: $X_1, X_2 \in \underline{X} \Rightarrow aX_1 + bX_2 \in \underline{X}$ for any real a and b.
     - Note that free portfolio formation rules out short sales constraints, bid/ask spreads, leverage limitations, etc.
  2. **Law of one price** : $P(aX_1 + bX_2) = aP(X_1) + bP(X_2)$ for any real a and b.
     - The law of one price says that investors cannot make instantaneous profits by repackaging portfolios.

Law of one price and existence of discount factor

Result

Given free portfolio formation and the law of one price, there exists a unique payoff $X^\star \in \underline{X}$ such that $P(X) = E(X^\star X)$ for all $X \in \underline{X}$

- Note that the existence of a discount factor implies the law of one price $E[M(X+Y)] = E[MX] + E[MY]$. The theorem reverses this logic.

Law of one price and existence of discount factor

Proof.

Riesz representation theorem: Assume that the payoff space $\underline{X}$ is generated by N basis payoffs ($\boldsymbol{X} = [X_1\ X_2\ ...\ X_N]'$ and $\underline{X} = \{\boldsymbol{c}'\boldsymbol{X}\}$). Let us find a discount factor $X^\star$ in the payoff space, i.e such that $X^\star = \boldsymbol{c}^{\star'}\boldsymbol{X}$. Construct $\boldsymbol{c}^\star$ such that it prices the basis assets. We want $\boldsymbol{P} = E(X^\star \boldsymbol{X}) = E(\boldsymbol{X}\boldsymbol{X}'c^\star)$. Take $\boldsymbol{c} = E(\boldsymbol{X}\boldsymbol{X}')^{-1}\boldsymbol{P}$. Linearity (law of one price) implies that $E(\boldsymbol{X}\boldsymbol{X}')$ is non singular. Thus, $X^\star = \boldsymbol{P}'E(\boldsymbol{X}\boldsymbol{X}')^{-1}\boldsymbol{X}$ is the discount factor in $\underline{X}$. It is a linear combination of $\boldsymbol{X}$ so it is in $\underline{X}$. It prices every X in $\underline{X}$. For every $X \in \underline{X}$ (thus for every $\boldsymbol{c}$), $E[X^\star \boldsymbol{X}'\boldsymbol{c}] = E[\boldsymbol{P}'E(\boldsymbol{X}\boldsymbol{X}')^{-1}\boldsymbol{X}\boldsymbol{X}'\boldsymbol{c}] = \boldsymbol{P}'\boldsymbol{c}$, and thus $P(\boldsymbol{c}'\boldsymbol{X}) = \boldsymbol{c}'\boldsymbol{P}(X)$. □

Law of one price and existence of discount factor

- Note that the theorem says there is a unique $X^\star$ in $\underline{X}$.
- There may be other discount factors not in $\underline{X}$.
- if markets are incomplete, there are an infinite number:

$$p = E[(M + \varepsilon)X]$$

for any ε to X

What does all this mean?

- note that any discount factor M that satisfies $E[MX] = P$ can be represented as:

$$M = X^* + \epsilon$$

with $E(\epsilon X) = 0$
- X^* is the projection of any SDF M on the space $\underline{X}$ of payoffs
- the pricing implications of any M for payoffs in $\underline{X}$ are the same as those of $proj(M|\underline{X})$:

$$p = E(MX) = E\left[\left(proj(M|\underline{X}) + \epsilon\right)x\right] = E\left[\left(proj(M|\underline{X})\right]\right.$$

Summary

- complete contingent claims: implies linearity of the pricing function and the law of one price
- incomplete markets: law of one price implies a linear pricing function, linearity implies existence of discount factor

No Arbitrage

Definition

Definition: Absence of arbitrage: A payoff space $\underline{X}$ and pricing function $P(X)$ leave no arbitrage opportunities if every payoff X that is always nonnegative ($X \geq 0$ almost surely), and strictly positive ($X > 0$) with some positive probability has some strictly positive price $P(X) > 0$.[a]

[a] In other words, no arbitrage says that you cannot get for free a portfolio that might pay off positively but will certainly never cost you anything.

SDF→No Arbitrage

- absence of arbitrage is a consequence of positive discount factors

Result

$P = E(MX)$ and $M(s) > 0$ imply no arbitrage.

Proof.

We have $M > 0$, $X \geq 0$ and in some states $X > 0$. Therefore, $E(MX) > 0$, thus $P(X) > 0$. □

No Arbitrage→SDF

Result

In complete markets, no arbitrage and the law of one price imply that there exists a unique $M > 0$ such that $P = E(MX)$.

Proof.

From LOP, $\exists X^{\star}$ such that $P(X) = E(X^{\star}X)$. In complete markets, the discount factor is unique. Suppose that $X^{\star} \leq 0$ for some states. Then form a payoff X that is 1 in those states and zero elsewhere. This payoff is strictly positive, but its price is negative, negating the assumption of no arbitrage. □

No Arbitrage→ SDF

Result

No arbitrage and the law of one price imply the existence of a strictly positive discount factor $M > 0$ such that $P = E(MX), \forall X \in \underline{X}$.

Proof.

Skip it □

- get intuition from $\Re^2$.
- Note that the theorem does not say that M is unique.
- the theorem says that a positive M exists, but it does not say that every M must be positive
- Assume that $\underline{X}$ is a subset of $\Re^S$. The theorem says that we can extend the pricing function defined on $\underline{X}$ to all possible payoffs in $\Re^S$ and not imply any arbitrage opportunities on that larger space of payoffs.

Roadmap

Projection Approach

- let Y be an unobserved SDF and postulate it has the following form:

$$Y = a + X'b + e$$

with e being orthogonal to X

- note that $X^{\star} = proj(Y|\underline{X}) = a + X'b$

Projection

- the slope coefficient b is derived from the population linear regression:

$$b = [cov(X, X)]^{-1} cov(X, Y)$$
$$a = EY - EX'b$$

with $cov(X, X) = E(XX') - E(X)E(X)'$

- data on X but no data on Y
- we do have data on $P = E(YX)$

Projection

- note that $P = E(YX)$ implies that :

$$cov(X,Y) = P - E(Y)E(X)$$

- this implies we can state:

$$b = [cov(X,X)]^{-1}(P - E(Y)E(X))$$

- once we have a guess for $E(Y)$, we can estimate b, the slope coefficient

Variance Bounds

- different derivation of SDF
- recall that:

$$Y = a + X'b + e$$

with e being orthogonal to X

- this implies that, because of the orthogonality, we have a bound on the variance :

$$var(Y) = var(X'b) + var(e)$$

- therefore:

$$(var(X'b))^{1/2} \leq \sigma(Y)$$

- HJ used the expression for b and the variance expression to trace out frontiers of admissible SDF's

Risk-free

- assume there is a risk-free asset
- note that returns are denoted $Z = X/P$
- if Z is constant, the return is risk-free
- this being the case, we can back out E(Y) from the average risk-free rate:

$$E(YZ^{rf}) = Z^{rf}E(Y) = 1$$

Excess Returns

- assume there is no risk-free asset and $E(Y)$ is not known
- suppose we observe excess returns $Z = X^s - X^b$:

$$E(YZ) = 0$$

- the slope coefficient is:

$$b = [cov(Z, Z)]^{-1}(0 - E(Y)E(Z))$$

- the variance bounds are given by:

$$\left(var(Z'b)\right) = E(Y)^2 E(Z)'[cov(Z, Z)^{-1}]E(Z)$$

- this implies that the bounds are given by:

$$\sigma(Y) \geq \left(E(Z)'[cov(Z, Z)^{-1}]E(Z)\right)^{.5} E(Y)$$

Returns

- this translates into an upper bound on $\frac{\sigma(Y)}{E(Y)}$

$$\frac{\sigma(Y)}{E(Y)} \geq \left(E(Z)'[cov(Z,Z)^{-1}]E(Z)\right)^{.5}$$

- if Z is scalar, then

$$\frac{\sigma(Y)}{E(Y)} \geq \frac{E(Z)}{\sigma(Z)}$$

- $\frac{\sigma(Y)}{E(Y)}$ is also referred to as the market price of risk

Returns

- consider returns: $P = E(PZ) = 1$
- in this case the slope coefficient is:

$$b = [cov(Z,Z)]^{-1}(1 - E(Y)E(Z))$$

- compute the bounds by computing b :

$$\sigma(Y) \geq \left(b'cov(Z,Z)b\right)^{.5}$$

Roadmap

- factor pricing models:

$$M_{t+1} = \boldsymbol{b}'\boldsymbol{f}_{t+1}$$

  - CAPM
  - ICAPM
  - APT
- imply beta representations

Beta Representation

Definition

The expected return-beta representation of a factor pricing model is

$$E(R^i) = \gamma + \beta_{i,a}\lambda_a + \beta_{i,b}\lambda_b + ..., \; i = 1, 2, ...N.$$

Note that the intercept γ is the same for all assets. The β terms are defined in time-series regressions of returns on factors:

$$R^i_t = a_i + \beta_{i,a}f_{a,t} + \beta_{i,b}f_{b,t} + ... + \varepsilon^i_t, \; t = 1, 2, ...T.$$

- we regress returns at $t+1$ on factors at $t+1$

Estimating the free Parameters

- to test the model, we can run a cross-sectional regression of average returns on the betas:

$$E_T(R^i) = \gamma + \widehat{\beta}_{i,a}\lambda_a + \widehat{\beta}_{i,b}\lambda_b + ... + \alpha_i, \; i = 1, 2, ...N.$$

  where $\widehat{\beta}_i$ are the right hand side variables and α_i are pricing errors

- only β's on the right hand side, not characteristics!

Beta Representation

- If there is a risk-free rate, its betas are all zeros, so $R^f = \gamma$.
- In terms of excess returns $R^{e,i}$, the expected return-beta representation has no constant:

$$E(R^{e,i}) = \beta_{i,a}\lambda_a + \beta_{i,b}\lambda_b + ..., \; i = 1, 2, ...N$$

- If the factors are returns, their betas are one on themselves and zero on all other factors. This implies that the market price of risk of traded factors is the mean of the factor:

$$E(f_a) = \lambda_a$$

CAPM

- the Capital Asset Pricing Model implies that expected excess returns satisfy:

$$E(R^{e,i}) = \beta_{i,M}\lambda_M$$

- look at the expected excess return on the market:

$$E(R^{e,M}) = \lambda_M$$

Fama-French

- two more risk factors:
  1. size (market cap): smb
  2. book-to-market: hml

Fama-French

- the Capital Asset Pricing Model implies that expected excess returns satisfy:

$$E(R^{e,i}) = \beta_{i,M}\lambda_M + \beta_{i,smb}\lambda_{smb} + \beta_{i,hml}\lambda_{hml}$$

- look at the expected excess return on *smb*:

$$E(R^{e,smb}) = \lambda_{smb}$$

Size and Book-to-Market

size // B/M	low	medium	high
small	1	2	3
big	4	5	6

Mean Variance Frontier

- Let R denote a vector of returns, with mean $\overline{R} = E(R)$ and variance-covariance matrix $\Sigma = E([R - \overline{R}][R - \overline{R}]')$.
- Let w denote the portfolio weights.
- Choose a portfolio to minimize variance for a given mean:

$$Min_w \; w'\Sigma w$$

subject to

$$w'\overline{R} = \mu$$

and subject to

$$w'1 = 1$$

Mean Variance Frontier

- Let 2λ and 2δ be the Lagrange multipliers.
- the first-order condition is:

$$\Sigma w - \lambda \overline{R} - \delta 1 = 0.$$

- this implies that the portfolio weights are:

$$w = \Sigma^{-1} \left(\lambda \overline{R} + \delta 1 \right)$$

Mean Variance Frontier

Result

As long as the variance-covariance matrix of returns is non-singular the mean-variance frontier exists: $w = \Sigma^{-1}(\lambda E + \delta 1)$.

Mean Variance Frontier

Plugging w in the constraints leads to:

$$\begin{aligned}\overline{R}'w &= \overline{R}'\Sigma^{-1}(\lambda\overline{R}+\delta 1) = \mu, \\ 1'w &= 1'\Sigma^{-1}(\lambda\overline{R}+\delta 1) = 1,\end{aligned}$$

or

$$\begin{aligned}\begin{bmatrix} \overline{R}'\Sigma^{-1}\overline{R} & \overline{R}'\Sigma^{-1}1 \\ 1'\Sigma^{-1}\overline{R} & 1'\Sigma^{-1}1 \end{bmatrix}\begin{bmatrix}\lambda \\ \delta\end{bmatrix} &= \begin{bmatrix}\mu \\ 1\end{bmatrix} \\ \begin{bmatrix} A & B \\ B & C \end{bmatrix}\begin{bmatrix}\lambda \\ \delta\end{bmatrix} &= \begin{bmatrix}\mu \\ 1\end{bmatrix},\end{aligned}$$

where $A = \overline{R}'\Sigma^{-1}\overline{R}$, $B = \overline{R}'\Sigma^{-1}1$ and $C = 1'\Sigma^{-1}1$. As a result:

$$\begin{aligned}\lambda &= \frac{C\mu - B}{AC - B^2} \\ \delta &= \frac{A - B\mu}{AC - B^2}\end{aligned}$$

Minimum Variance

Result

The portfolio weights are for a given mean return μ are:

$$w = \Sigma^{-1}\frac{\overline{R}(C\mu - B) + 1(A - B\mu)}{AC - B^2}.$$

The minimum variance portfolio is:

$$w = \frac{\Sigma^{-1}1}{1'\Sigma 1}.$$

Definitions

- $R^\star$ is the return corresponding to the payoff that can be the SDF
- Let us define $R^\star$ and $R^{e\star}$ as:

$$\begin{aligned} R^\star &\equiv \frac{X^\star}{P(X^\star)} = \frac{X^\star}{E(X^{\star 2})}, \\ R^{e\star} &\equiv proj(1|\underline{R^e}), \text{ where } \underline{R^e} \text{ is the space of excess returns.} \end{aligned}$$

What is $R^{e\star}$

- $R^{e,*}$ represents means the same way X^* represent prices
- analogy between X^* and $R^{e,*}$:

$$P(X) = E(MX) = E\left[proj(M|\underline{X})X\right] = E(X^\star X)$$

$$E(R^e) = E(1 \times R^e) = E\left[proj(1|\underline{R^e})X\right] = E(R^{e\star}R^e)$$

Decomposition Result

Result

Every return R^i can be expressed as

$$R^i = R^\star + w^i R^{e\star} + n^i,$$

where w^i is a number, and n^i is an excess return such that $E(n^i) = 0$. The three components are orthogonal: $E(R^\star R^{e\star}) = E(R^\star n^i) = E(R^{e\star} n^i) = 0$.

Mean Variance Frontier

Result

R^{mv} is on the mean-variance frontier if and only if $R^{mv} = R^\star + wR^{e\star}$ for some number w.

- we can *trace out the mean variance frontier* simply by varying w
- two fund theorem

Proof of Decomposition

Proof.

We know that $E(R^\star R^{e\star}) = E(X^\star R^{e\star})/E(X^{\star 2}) = 0$. Pick any w^i and define $n^i = R^i - R^\star - w^i R^{e\star}$. n^i is an excess return, so it is orthogonal to $R^\star$ and $E(n^i) = E(R^{e\star} n^i)$. Therefore, $R^{e\star}$ is orthogonal to n^i if and only if we pick w^i so that $E(n^i) = 0$. Since $E(n^i) = 0$ and the three components are orthogonal, we obtain:

$$\begin{aligned} E(R^i) &= E(R^\star) + w^i E(R^{e\star}), \\ \sigma^2(R^i) &= \sigma^2(R^\star + w^i R^{e\star}) + \sigma^2(n^i). \end{aligned}$$

Returns with $n^i = 0$ minimize variance for each mean. □

Construction

- How to construct $R^\star$ and $R^{e\star}$ from basic assets? First recall that $X^\star = \boldsymbol{P}' E(\boldsymbol{X}\boldsymbol{X}')^{-1}\boldsymbol{X}$. Thus,

$$R^\star = \frac{X^\star}{P(X^\star)} = \frac{P' E(\boldsymbol{X}\boldsymbol{X}')^{-1}\boldsymbol{X}}{\boldsymbol{P}' E(\boldsymbol{X}\boldsymbol{X}')^{-1}\boldsymbol{P}}.$$

Projecting the vector one on the space of excess returns:

$$R^{e\star} = \frac{E(R^e)'}{E(R^e R^{e'})^{-1}} R^e,$$

where R^e is the vector of basis excess returns.

Construction

- in mean/standard deviation space, lines of constant second moments are circles
- R^* is the minimum second moment return: is the intersection between the smallest circle and the mean/variance frontier
  - for any return:
$$R = R^* + wR^{e*} + n$$
$$E(R^2) = E(R^{*2} + w^2 R^{e*2} + E(n^2)$$
  - set $n = 0$ and $w = 0$ to get the smallest second moment
- R^* is on the inefficient segment of the mean-variance frontier

Mean-Variance Frontiers and SDF's

- connection between mean-variance frontiers and SDF's:

$$\min_{M \text{ that price } X \in \underline{X}} \frac{\sigma(M)}{E(M)} = \max_{\text{excess returns } R^e \in \underline{X}} \frac{E(R^e)}{\sigma(R^e)}$$

- From $E(MR^e) = 0$, we obtained:

$$\frac{\sigma(M)}{E(M)} \geq \frac{|E(R^e)|}{\sigma(R^e)}.$$

- Graphical construction of Hansen-Jagannathan bounds.

- Algebraic argument:

$$M = \underbrace{E(M) + [\mathbf{P} - E(M)E(\mathbf{X})]'\mathbf{\Sigma}^{-1}[\mathbf{X} - E(\mathbf{X})]}_{X^\star} + \epsilon,$$

where $\mathbf{\Sigma} = cov(\mathbf{X}, \mathbf{X})$ and $E(\epsilon) = 0,\ E(\epsilon\mathbf{X}) = \mathbf{0}$ (regression of any discount factor on the space of payoffs). Since $\sigma(\epsilon) > 0$, the Hansen-Jagannathan bound is:

$$\sigma^2(M) \geq [\mathbf{P} - E(M)E(\mathbf{X})]'\Sigma^{-1}[\mathbf{P} - E(M)E(\mathbf{X})].$$

- Decomposition of any discount factor $M = X^\star + \epsilon$:

$$M = X^\star + wE^\star + n,$$

where $E^\star$ is defined as:

$$E^\star \equiv 1 - proj(1|\underline{X}) = proj(1|E).$$

Recall that $X^\star = \boldsymbol{P}'E(\boldsymbol{X}\boldsymbol{X}')^{-1}\boldsymbol{X}$. We can compute $E^\star$ as:

$$E^\star = 1 - E(\boldsymbol{X})'E(\boldsymbol{X}\boldsymbol{X}')^{-1}\boldsymbol{X}.$$

And the variance minimizing discount factors are:

$$\begin{aligned} M^\star = X^\star + wE^\star &= \boldsymbol{P}'E(\boldsymbol{X}\boldsymbol{X}')^{-1}\boldsymbol{X} + w(1 - E(\boldsymbol{X})'E(\boldsymbol{X}\boldsymbol{X}')^{-1}\boldsymbol{X}) \\ &= w + [\boldsymbol{P} - wE(\boldsymbol{X})]'E(\boldsymbol{X}\boldsymbol{X}')^{-1}\boldsymbol{X}. \end{aligned}$$

As a result:

$$\begin{aligned} E(M^\star) &= w + [\boldsymbol{P} - wE(X)]'E(\boldsymbol{X'})^{-1}E(\boldsymbol{X}), \\ \sigma^2(M^\star) &= [\boldsymbol{P} - wE(\boldsymbol{X})]'E(\boldsymbol{X}\boldsymbol{X}')^{-1}[P - wE(\boldsymbol{X})]. \end{aligned}$$

- As a result:

$$\begin{aligned} E(M^\star) &= w + [\boldsymbol{P} - wE(\boldsymbol{X})]'E(\boldsymbol{X}\boldsymbol{X}')^{-1}E(\boldsymbol{X}), \\ \sigma^2(M^\star) &= [\boldsymbol{P} - wE(\boldsymbol{X})]'E(\boldsymbol{X}\boldsymbol{X}')^{-1}[\boldsymbol{P} - wE(\boldsymbol{X})]. \end{aligned}$$

Duality

Hansen-Jagannathan frontiers are equivalent to mean-variance frontiers.

Result

Hansen-Jagannathan frontiers are equivalent to mean-variance frontiers. As with returns, we can trace out the frontier by varing w in

$$M = X^{\star} + wE^{\star}$$

,

HJ Bounds can do better

- we could impose positivity:

$$\min \sigma^2(M)$$

such that

$$P = E(MX), M > 0,$$

for fixed $E(M)$