# Lecture 3: Beta Representation and Conditioning Information

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# Roadmap

1. SDF, betas and frontiers
   - From discount factors to beta representation
   - From mean-variance frontier to SDF and $\beta$
   - Factor models and discount factors
2. What have we learned?
3. Conditioning information
   - Scaled payoffs
   - Instruments and Managed Portfolios
   - Jagannathan and Wang
   - Lettau and Ludvigson
   - Conditional and unconditional moments
   - Scaled factors





# Beta representation using $M$

### Result

*The basic asset pricing equation*

$$P = E(MX)$$

*implies*

$$E(R^i) = \gamma + \beta_{i,M}\lambda_M.$$

$\lambda_M$ *is the expected excess return on an asset that has a loading of one on the SDF. Note that this is an insurance contract.*

Proof.

Here is the derivation:

$$E(R^i) = \frac{1}{E(M)} - \frac{cov(M,R^i)}{E(M)} = \frac{1}{E(M)} + \underbrace{\frac{cov(M,R^i)}{var(M)}}_{\beta_{i,M}} \underbrace{-\frac{var(M)}{E(M)}}_{\lambda_M}.$$

The risk price is negative, because this asset with a loading of one is an insurance contract. □





## Beta Representation

1. we can also use $X^\star$ as a factor
2. we can also use $R^\star$ as a factor
   - when using a return as factor, the risk price is the average return ✓

# Beta representation using $X^\star$ and $R^\star$

**Result**

*The basic asset pricing equation implies that:*

$$E(MR^i) = 1$$

*implies an expected return-beta model with* $X^\star = proj(M|\underline{X})$ *or* $R^\star = X^\star / E(X^{\star 2})$ *as factors, where*

$$E(R^i) = \gamma + \beta_{R^i,X^\star}\lambda_{X^\star}$$

*and*

$$E(R^i) = \gamma + \beta_{R^i,R^\star}\underbrace{[E(R^\star) - \gamma]}_{\text{risk premium}}.$$





# Beta representation using $X^\star$

**Proof.**

Recall that $P = E(MX)$ implies $P = E(proj(M|\underline{X})X) = E(X^\star X)$. Then, from $E(X^\star R^i) = 1$:

$$E(R^i) = \frac{1}{E(X^\star)} - \frac{cov(X^\star, R^i)}{E(X^\star)} = \frac{1}{E(X^\star)} - \frac{cov(X^\star, R^i)}{var(X^\star)}\frac{var(X^\star)}{E(X^\star)}.$$

□

# Beta representation using $R^\star$

Proof.

To derive the equivalent expression in terms of returns: start from $R^\star = X^\star / E(X^{\star 2})$, multiply by $R^\star$, take expectations: $E(R^{\star 2}) = 1/E(X^{\star 2})$, and thus, $E(X^\star) = E(R^\star)/E(R^{\star 2})$, then $E(X^\star) = E(R^\star)/E(R^{\star 2})$. As a result:

$$E(R^i) = \frac{E(R^{\star 2})}{E(R^\star)} - \frac{cov(R^\star, R^i)}{E(R^\star)} = \frac{E(R^{\star 2})}{E(R^\star)} - \frac{cov(R^\star, R^i)}{var(R^\star)} \frac{var(R^\star)}{E(R^\star)},$$

$$E(R^i) = \gamma + \beta_{R^i, R^\star} \lambda_{R^\star}.$$

We can apply the above equation to $R^\star$ itself: $E(R^\star) = \gamma - \frac{var(R^\star)}{E(R^\star)}$. And thus, the beta-model is:

$$E(R^i) = \gamma + \beta_{R^i, R^\star} [E(R^\star) - \gamma].$$

□





# Mean-variance frontier

Result

*There is a discount factor of the form $M = a + bR^{mv}$ if and only if $R^{mv}$ is on the mean-variance frontier or equivalently*

$$R^{mv} = R^\star + wR^{e\star},$$

*and $R^{mv}$ is not the risk-free rate.*

- Note: when there is no risk-free rate, the condition is that $R^{mv}$ is not the constant-mimicking portfolio return.

## Mean-variance frontier

**Proof.**

Pick an arbitrary $R$. Try $M = a + bR = a + b(R^\star + wR^{e\star} + n)$. Determine $a$ and $b$ to make the model price $R^\star$ and $R^{e\star}$:

$$
\begin{aligned}
1 &= E(MR^\star) = aE(R^\star) + bE(R^{\star 2}), \\
0 &= E(MR^{e\star}) = aE(R^{e\star}) + bwE(R^{e\star 2}) = (a + bw)E(R^{e\star}).
\end{aligned}
$$

Recall that $R^{e\star} = proj(1|\underline{R^e})$, thus $E(R^e) = E(R^{e\star}R^e)$, which is also true for $R^e = R^{e\star}$, and as a result, $E(R^{e\star}) = E(R^{e\star 2})$. Solve for $a$ and $b$:

$$
\begin{aligned}
a &= \frac{w}{wE(R^\star) - E(R^{\star 2})}, \\
b &= -\frac{1}{wE(R^\star) - E(R^{\star 2})}.
\end{aligned}
$$

□





## Mean-variance frontier

**Proof.**

Test whether this discount factor $M$ prices any payoff $X^i \in \underline{X}$: $X^i = y^i R^\star + w^i R^{e\star} + n^i$. The price of $X^i$ is $y^i$ since both $R^{e\star}$ and $n^i$ are excess returns. But:

$$
\begin{aligned}
E(MX^i) &= E[\frac{(w - R^\star - wR^{e\star} - n)(y^i R^\star + w^i R^{e\star} + n^i)}{wE(R^\star) - E(R^{\star 2})}], \\
&= \frac{wy^i E(R^\star) - y^i E(R^{\star 2}) - E(nn^i)}{wE(R^\star) - E(R^{\star 2})}.
\end{aligned}
$$

To get $P(X^i) = y^i$, we need $E(nn^i) = 0, \forall X^i \in \underline{X}$, thus $n = 0$ and $R$ is on the mean-variance frontier. □

## Mean-variance frontier

- This construction does not work if $w = E(R^{\star 2})/E(R^\star) = 1/E(X^\star)$.
- If there is a risk-free rate, $R^f = 1/E(X^\star)$ and we are ruling out the case $R = R^\star + R^f R^{e\star}$, which is the risk-free rate. To see this, recall that:

$$R^{e\star} = 1_V - proj(1_V | R^\star) = 1_V - \frac{E(R^\star)}{E(R^{\star 2})} R^\star = 1_V - \frac{1}{R^f} R^\star.$$

- If there is no risk-free rate, $R^\star + E(R^{\star 2})/E(R^\star) R^{e\star}$ is interpreted as a 'constant-mimicking portfolio return'.





## Summary

- the existence of $P = E(MX)$ implies a single beta representation ✓
- beta pricing models are equivalent to linear factor models for the discount factor

# Factor Models

**Result**

*Given the model:*

$$M = 1 + [f - E(f)]'b \text{ and } E(MR^e) = 0, \quad (1)$$

*one can find $\lambda$ such that:*

$$E(R^e) = \beta'\lambda, \quad (2)$$

*where $\beta$ are the multiple regression coefficients of excess returns $R^e$ on the factors. Conversely, given $\lambda$ in (2), one can find $b$ such that (1) holds.*





# Factor Models

**Result**

*Given the model:*

$$M = a + b'f \text{ and } E(MR^i) = 1, \quad (3)$$

*one can find $\gamma$ and $\lambda$ such that:*

$$E(R^i) = \gamma + \lambda'\beta_i, \quad (4)$$

*where $\beta_i$ are the multiple regression coefficients of returns $R^i$ on the factors. Conversely, given $\gamma$ and $\lambda$ in (11), one can find $a$ and $b$ such that (3) holds.*

# Factor Models

**Proof.**

Start with $M = a + b'f$ and $E(MR^i) = 1$. Rewrite $M$ such that $E(f) = 0$. Then:

$$E(R^i) = \frac{1}{E(M)} - \frac{cov(M, R^i)}{E(M)} = \frac{1}{a} - \frac{E(Rf')b}{a}.$$

Define $\beta_i$ as regression coefficients: $\beta_i = E(ff')^{-1}E(fR^i)$ to get:

$$E(R^i) = \frac{1}{a} - \beta_i' \frac{E(ff')b}{a}$$

Now define $\gamma$ and $\lambda$ as:

$$\gamma \equiv \frac{1}{E(M)} = \frac{1}{a},$$
$$\lambda \equiv -\frac{cov(f, f')b}{a} = -\gamma E(Mf).$$

. □





# Non-traded Factors

- a test of $\lambda = 0$ is a test of whether the 'factor is priced'
- $\lambda = -\gamma E(Mf)$ is the price of the demeaned factor moved forward at the risk-free rate
- if the factor is not traded, it's a shadow price

## Traded Factors

- if the factors are traded returns, it's the factor risk premium:

$$\lambda = -\gamma E[Mf] = -\gamma E\left[M(\widetilde{f} - E[\widetilde{f}])\right]$$
$$= -\gamma \left(1 - \frac{E(\widetilde{f})}{\gamma}\right)$$

- the factor risk premium is the expected return on the factor less $\gamma$:

$$\lambda = E(\widetilde{f}) - \gamma$$





## Factors

- factors need not be returns
- factors need not be orthogonal
- factors need not be serially uncorrelated

## Factor Mimicking Payoff

**Definition**

A factor-mimicking payoff is $f^\star = proj(f|\underline{X})$, a factor-mimicking return is $f^\star = proj(f|\underline{X})/P[proj(f|\underline{X})]$.





## Factor Mimicking Payoff

- the factor mimicking payoffs have the same pricing implications:

$$P = E[MX] = E[b'fX] = E[b' proj(f|\underline{X})X] = E[b'f^\star X]$$

- more common to use factor-mimicking returns:

$$f^\star = proj(f|\underline{X})/P[proj(f|\underline{X})]$$

- if all test-assets are excess returns, then: $f^\star = proj(f|\underline{R^e})$
- a linear combination of the factors is on the mean-variance frontier

$$X^\star = proj[M|\underline{X}] = proj[b'f|\underline{X}] = b' proj[f|\underline{X}] = b'f^\star$$

  and hence the minimum second moment portfolio

$$R^\star = \frac{X^\star}{E[(X^\star)^2]}$$

  is linear in $f^\star/P[f^\star]$

# Single Factor Representation

**Result**

*There is a single beta representation with a return $R^{mv}$ as a factor (Roll 1976, Hansen and Richards, 1987):*

$$E[R^i] = \gamma_{R^{mv}} + \beta_{i,R^{mv}} \left[ E(R^{mv} - \gamma_{R^{mv}}) \right]$$

*if and only if the return is mean-variance efficient and it is not the minimum-variance return.*

- before, single factor beta representations had only been derived in the context of specific, fully specified models (CAPM)
- this shows there always exists a single factor representation





# Single Factor Representation

**Proof.**

The mean-variance frontier is $R^{mv} = R^\star + wR^{e,\star}$. Any return can be described as $R^i = R^\star + w^i R^{e,\star} + n^i$. This implies that the expected return on asset $i$ is given by:

$$E[R^i] = R^\star + w^i R^{e,\star}.$$

Once we know $w^i$, we 're done and we can find the single factor beta representation by plugging $w^i$ back into the equation for $E(R^i)$. Solve for $w^i$ from:

$$\begin{aligned} cov\left(R^i, R^{mv}\right) &= cov\left(R^\star + w^i R^{e,\star}, R^\star + wR^{e,\star}\right) \\ &= var(R^\star) - wE(R^\star)(R^\star) \\ &+ w^i \left[ wvar(R^{e,\star}) - E(R^\star)E(R^{e,\star}) \right] \end{aligned}$$

□

# Roadmap







# it's about the Economics!

- there *always* is a single-beta representation since there always is a mean-variance efficient frontier
- the model delivers a specific return that is mean-variance efficient (e.g. CAPM: Market return)
  - key question: how do we define the reference portfolio?
- $P = E[MX]$ is equivalent to Roll's theorem
- all the work is in writing down the discount factor model:

$$M = f(data)$$

## Fishing Expeditions

- we can always construct a SDF that prices payoffs exactly from the sample moments:

$$P(X) = E_T[X^\star X]$$

  if the sample variance-covariance matrix is non-singular.
- caveat for ad hoc pricing models:
  - only reason they do not work perfectly is because of restrictions on the number of factors
  - it's all about the restrictions!
- what's wrong with finding an ex post mean-variance efficient portfolio that prices assets by construction? it won't work in the next sample





## Which factors

- macro-economic factors are measured with error
- returns are not
- don't run horse races between models with macro-economic factors and returns (it's a bit silly)
- for many purposes, it might be better to use the factor mimicking portfolio of say aggregate consumption growth, rather than aggregate consumption growth

# Roadmap







# Conditioning

- prices reflects lots of information that the econometrician is not privy too
- we want to go from conditional moments to unconditional moments that we can test without making arbitrary assumptions about the information set of the investor: add *managed portfolios*
  - avoid making assumption about conditional distribution of asset prices

- Scaled payoffs and managed portfolios:

$$P_t = E_t(M_{t+1}X_{t+1}) \Rightarrow E(P_tZ_t) = E(M_{t+1}X_{t+1}Z_t).$$

  - Think size, beta, industry, book/market portfolios.
- Checking the expected price of all managed portfolios is, in principle, sufficient to check all the implications of conditioning information:

$$E(P_tZ_t) = E(M_{t+1}X_{t+1}Z_t), \forall Z_t \in I_t \Rightarrow P_t = E(M_{t+1}X_{t+1}|I_t).$$

- law of iterated expectations
- Choice of instruments $Z_t$: only variables that forecasts returns or $M$ (or their higher moments and co-moments) add any information.





## Managed Portfolio

- multiply the price by any instrument $Z_t$:

$$Z_tP_t = E_t\left[M_{t+1}X_{t+1}Z_t\right]$$

- now take unconditional expectations:

$$E(Z_tP_t) = E\left[M_{t+1}X_{t+1}Z_t\right]$$

- another implication of the model
- *managed* portfolios:
  - simply define new payoff $X = X_tZ_t$ and new price $P = Z_tP_t$
  - all we're saying is $P = E[MX]$
  - linear investment rule

# Back to Testing Unconditional Moments

- to test conditional moments
  - construct managed portfolio payoffs
  - test unconditional moments
- is this enough? yes.
- note that

$$E\left[(M_{t+1}X_{t+1} - P_t)Z_t\right] = 0$$

for every $z_t \in I_t$ implies

$$E\left[M_{t+1}X_{t+1} - P_t | I_t\right] = 0$$





# Models

- does it matter whether we look at conditional vs unconditional moments?
  - no: explicit discount factors with constant parameters: CCAPM
  - yes: discount factors with time-dependent parameters: CAPM

# Consumption-CAPM

- consider the consumption-CAPM
- the SDF:

$$M_{t+1} = \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma}$$

- note that the parameters do not depend on any conditioning information!





# Consumption-CAPM

- linear model? use first-order log-linear approximation:

$$\frac{M_t}{E[M_t]} \approx 1 + m_t - E[m_t]$$

- this produces:

$$\frac{M_t}{E[M_t]} \approx 1 - \gamma \left( \Delta c_{t+1} - E[\Delta c_{t+1}] \right)$$

- from the unconditional Euler equation:

$$E[R^i_{t+1} - R^f_t] = \gamma Cov \left[ R^i_{t+1} - R^f_t, \Delta c_{t+1} - E[\Delta c_{t+1}] \right]$$

# Conditional CAPM

- consider the CAPM:

$$M_{t+1} = a - bR_{w,t+1}$$

  - model should price any two assets accurately
  - price the risk-free rate and the market:

$$\begin{aligned} 1 &= E_t(M_{t+1}R_{w,t+1}) \\ 1 &= E_t(M_{t+1})R_t^f \end{aligned}$$





# Conditional CAPM

- consider the CAPM:

$$M_{t+1} = a - bR_{w,t+1}$$

  - these two equations immediately imply that:

$$a = \frac{1}{R_t^f} + bE_t(R_{w,t+1})$$

$$b = \frac{E_t(R_{w,t+1}) - R_t^f}{R_t^f \sigma_t^2(R_{w,t+1})}$$

# Conditional CAPM

- if the market returns are predictable or the risk-free rate varies over time, then the coefficients change!

$$M_{t+1} = a_t - b_t R_{w,t+1}$$

- note that

$$E_t \left[ \left( a_t - b_t R_{w,t+1} \right) R_{t+1} \right] = 1$$

does **not** imply that:

$$E \left[ \left( a - b R_{w,t+1} \right) R_{t+1} \right] = 1$$





# Conditional Expected Return-Beta Model

- same caveat applies to the beta representation!
- also note that

$$E_t(R^i_{t+1}) = R^f_t + \lambda'_t \beta^i_t,$$

does **not** imply that:

$$E(R^i) = \gamma + \lambda' \beta_i,$$

# Conditional Expected Return-Beta Model

- even if $\lambda_t$ is constant
- average $\beta^i_t$ is not $\beta^i$ !!
  - *average of conditional covariance is not unconditional covariance*





# Jagannathan and Wang

- start with conditional CAPM:

$$E_t(R^i_{t+1}) = \gamma_{0,t} + \lambda_t' \beta^i_t.$$

**Result**

*The conditional CAPM implies that the unconditional expected return equals:*

$$E(R^i) = \overline{\gamma}_0 + \overline{\lambda}\overline{\beta}^i + Cov\left(\lambda_t, \beta^i_t\right),$$

*with* $\overline{\gamma}_0 = E[\gamma_{0,t}]$ *and* $\overline{\lambda} = E[\lambda_t]$ *and* $\overline{\beta}^i = E[\beta^i_t]$

- note: the expected $\beta$ is not the unconditional $\beta$

# Jagannathan and Wang

- correlation between $\beta$ and the risk premium $\lambda_t$: **non-zero**
  - more leveraged firms have higher conditional betas
- project the conditional betas on the conditional risk premium:

$$\beta_t^i = \overline{\beta}_i + \theta_i(\lambda_t - \overline{\lambda}) + \eta_t^i$$

  with $\theta_i$ as the *beta premium sensitivity*
- substitute back into expect return equation:

$$E(R^i) = \overline{\gamma}_0 + \overline{\lambda\beta}^i + \theta_i var(\lambda_t),$$





# Unconditional Two-Beta Model

**Result**

*J-W show this model implies un* ***unconditional beta*** *representation (two-beta representation):*

$$E(R^i) = a_0 + a_1\beta^i + a_2\beta^{i,\lambda},$$

*with* $\beta^i = \frac{cov(R_{i,t},R_{w,t})}{var(R_{w,t})}$ *and* $\beta^{i,\lambda} = \frac{cov(R_{i,t+1},\lambda_t)}{var(\lambda_t)}$

- two-beta unconditional beta model starting from *single beta* specification
- different from multi-beta models (Ross' APT)

## Risk Price

- to implement the model, JW assume the market risk premium is linear in the corporate default spread:

$$\lambda_t = \kappa_0 + \kappa_1 R_t^{prem}$$

- they want to capture counter-cyclical variation in the risk price





## Unconditional Two-Beta Model

**Result**

*J-W show this model implies un* ***unconditional beta*** *representation (two-beta representation):*

$$E(R^i) = c_0 + c_m \beta^i + c_{prem} \beta^{i,prem},$$

*with* $\beta^i = \frac{cov(R_{i,t}, R_{w,t})}{var(R_{w,t})}$ *and* $\beta^{i,prem} = \frac{cov(R_{i,t+1}, R_t^{prem})}{var(R_t^{prem})}$

# Conditional Consumption CAPM

- consider the conditional Consumption-CAPM: (conditional linear factor model)

$$M_{t+1} = a_t - b_t \Delta c_{t+1}$$

- assume linearity:

$$a_t = \gamma_0 + \gamma_1 z_t$$

- assume linearity:

$$b_t = \eta_0 + \eta_1 z_t$$





# Conditional Consumption CAPM

- substitute back into expression for $M_{t+1}$

$$M_{t+1} = \gamma_0 + \gamma_1 z_t - (\eta_0 + \eta_1 z_t)\Delta c_{t+1}$$

- scaled multi-factor model with constant coefficients
- the vector of factors:

$$F_{t+1} = (1, z_t, \Delta c_{t+1}, z_t \Delta c_{t+1})$$

and

$$M_{t+1} = c' F_{t+1}$$

- Lettau and Ludvigson propose $cw_t$ as instrument $z_t$

# Unconditional Multi-factor Beta Representation

**Result**

$$E[R_{t+1}] = \gamma_0 + \lambda' \beta.$$

*where $\beta^i$ is a vector of regression coefficients of of returns on the multiple factors.*

- not a two factor beta model, because there is a cross-term!





# Motivation for Conditional Consumption CAPM

- why time varying $a$ and $b$ coefficients?
- consider Campbell and Cochrane's habit model:

$$M_{t+1} = \delta \left[1 - \gamma\lambda(s_t) - \gamma(\phi - 1)(s_t - s) - \gamma[1 + \lambda(s_t)]\Delta c_{t+1}\right]$$

  - $X_t$ is the external habit
  - the surplus consumption ratio is $S_t \equiv \frac{C_t - X_t}{C_t}$
  - $g$ is the mean consumption growth rate
  - $\lambda(s_t)$ is the sensitivity function

### Result

*A conditional factor model does not imply a fixed-weight or unconditional factor model:*

- $M_{t+1} = b_t' f_{t+1},\ P_t = E_t(M_{t+1} X_{t+1})$ *does not imply that* $\exists b /\ M_{t+1} = b f_{t+1},\ P_t = E_t(M_{t+1} X_{t+1})$
- $E_t(R_{t+1}) = \beta_t' \lambda_t$ *does not imply* $E(R_{t+1}) = \beta' \lambda$
- *Conditional mean-variance efficiency does not imply unconditional mean-variance efficiency.*
- *The converse statements are true, if managed portfolios are included.*





### Proof.

Start from $M = a_t + b_t f_{t+1}$. Then:

$$\begin{aligned} 1 &= E[(a_t + b_t f_{t+1}) R_{t+1}], \\ &= E[a_t R_{t+1} + b_t f_{t+1} R_{t+1}], \\ &= E(a_t) E(R_{t+1}) + E(b_t) E(f_{t+1} R_{t+1}) \\ &+ cov(a_t, R_{t+1}) + cov(b_t, f_{t+1} R_{t+1}). \end{aligned}$$

Thus, the unconditional model only holds if the covariance terms are zero. □

### Result

*Hansen-Richard (1987) critique: Many models, such as the CAPM, imply a conditional factor model, but we test them by minimizing errors on average portfolio returns. These conditional models may be true and fail these tests, because a conditional model does not imply an unconditional one. We do not know the entire investors' information set; we condition down using the variables we observe. Thus, a conditional factor model is inherently not testable.*





- Expand the set of factors to test conditional factor models: $factors = f_{t+1} \bigotimes z_t$.
- Example:

$$\begin{aligned} M_{t+1} &= a(z_t) + b(z_t) f_{t+1}, \\ &= a_0 + a_1 z_t + (b_0 + b_1 z_t) f_{t+1}, \\ &= a_0 + a_1 z_t + b_0 f_{t+1} + b_1 (z_t f_{t+1}). \end{aligned}$$