Lecture 6: Estimation of Linear Factor Models

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January, 2009

http://www.anderson.ucla.edu/x18932.xml

Outline

1. Linear Factor Models
2. GMM estimation of Linear Factor Models
3. Regression-Based Tests of Linear Factor Models
   - Simple Tests
   - Tests without Distributional Assumptions
   - Cross-sectional Regressions
   - Fama-McBeth Procedure
4. Conditional CAPM and CCAPM
   - Conditional Consumption CAPM
   - Conditional Human Capital CAPM
   - Lewellen and Nagel Critique

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Beta Representation

- example of a linear factor model: 3-factor Fama-French model

$$\boldsymbol{f} = \left[\begin{array}{ccc} R_t^m & R_t^{smb} & R_t^{hml} \end{array} \right]$$

- this model implies a beta representation of expected excess returns:

$$E[R^{i,e}] = \beta_i^m \lambda^m + \beta_i^{smb} \lambda^{smb} + \beta_i^{hml} \lambda^{hml}$$

Equivalent Representation

- suppose we are using excess return data $\{\boldsymbol{R}_t^e\}$
- equivalent (in lieu of beta rep) representation is given by:

$$E[MR^{i,e}] = 0$$

where the linear factor model is given by:

$$M = 1 - \boldsymbol{b}'\boldsymbol{f}$$

  - note that the constant was normalized to 1 (cannot be estimated off excess return data)
- this is a moment condition that we can use in GMM estimation

Different Estimation Methods for Linear Factor Models

- GMM
- OLS
- Fama-MacBeth

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Moment Conditions

- the moment conditions are given by:

$$g_T(\boldsymbol{b}) = E_T(M\boldsymbol{R}^e) = E_T(\boldsymbol{R}^e) - E_T(\boldsymbol{R}^e\boldsymbol{f}')\boldsymbol{b}$$

- d is defined as:

$$\boldsymbol{d} = \frac{\partial g_T(\boldsymbol{b})}{\partial \boldsymbol{b}'} = E_T(\boldsymbol{f}\boldsymbol{R}^{e\prime}).$$

- The first-order condition of $min_b\ \boldsymbol{g}_T(\boldsymbol{b})'W\boldsymbol{g}_T(\boldsymbol{b})$ is:

$$\boldsymbol{d}'W[E_T(\boldsymbol{R}^e) + \boldsymbol{b}'\boldsymbol{d}] = 0.$$

Cross-sectional regression

Result

The GMM estimates are cross-sectional regression coefficients of mean excess returns on the second moment of returns with factors:

$$\hat{b}_1 = (\boldsymbol{d}'\boldsymbol{d})^{-1}\boldsymbol{d}'E_T(\boldsymbol{R}^e),$$
$$\hat{b}_2 = (\boldsymbol{d}'S^{-1}\boldsymbol{d})^{-1}\boldsymbol{d}'S^{-1}E_T(\boldsymbol{R}^e).$$

- because of the linearity, we can obtain a simple expression for the GMM estimator

Beta Representation

- Assume mean zero factors:

$$E(M'\boldsymbol{R}^e) = E[(1-\boldsymbol{b}'\boldsymbol{f})R^e] = 0,$$
$$E(R^e) = \boldsymbol{b}'cov(R^e,\boldsymbol{f}) = b'E(\boldsymbol{f}'\boldsymbol{f})\frac{cov(R^e,\boldsymbol{f})}{E(\boldsymbol{f}'\boldsymbol{f})} = \boldsymbol{\lambda}'\boldsymbol{\beta}.$$

  λ_j captures whether factor f_j is priced: $\lambda = \boldsymbol{b}'E[\boldsymbol{f}'\boldsymbol{f}] = -E(M\boldsymbol{f})$.
- The distinction between b and λ only matters when the factors are correlated.
- When the factors are orthogonal, $E(\boldsymbol{f}\boldsymbol{f}')$ is diagonal, and each $\lambda_j = 0$ if and only if $b_j = 0$.

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Single Factor Model

- the model :

$$E[R^{e,i}] = \beta_i E[f]$$

  where the factor is an excess return
- the model applies to the factor itself:

$$E[f] = 1 \times \lambda$$

- The betas are defined by **OLS time series regression coefficients**:

$$R_t^{e,i} = \alpha_i + \beta_i f_t + \varepsilon_t^i.$$

  - note that the regression intercepts α_i should be zero
  - the regression intercepts are the pricing errors

Testing

- we assume that there is
  1. no autocorrelation in ε_t^i
  2. no heteroscedasticity in ε_t^i
- tests are asymptotically valid even with stochastic regressors

Testing the Model: Single Factor

Result

We can test the model

$$E(R^{e,i}) = \beta_i E[f]$$

by running time series regressions:

$$R_t^{ei} = \alpha_i + \beta_i f_t + \varepsilon_t^i, t = 1, 2, \ldots, T$$

With i.i.d errors, homoscedasticity, and independence of the factors, the test statistic for the pricing errors:

$$T\left[1 + \frac{E_T(f)^2}{var_T(f)}\right]^{-1} \boldsymbol{\alpha}'\Sigma^{-1}\boldsymbol{\alpha} \sim \chi_N^2.$$

Testing the Model: Multiple Factors

Result

We can test the model

$$E(R^{e,i}) = \boldsymbol{\beta}_i' E[\boldsymbol{f}]$$

by running time series regressions:

$$R_t^{ei} = \alpha_i + \boldsymbol{\beta}_i' \boldsymbol{f}_t + \varepsilon_t^i, t = 1, 2, \ldots, T$$

With i.i.d errors, homoscedasticity, and independence of the factors, the test statistic for the pricing errors:

$$(T - N - K)/N \left[1 + E_T(\boldsymbol{f})' \Omega^{-1} E_T(\boldsymbol{f})\right]^{-1} \boldsymbol{\alpha}' \Sigma^{-1} \boldsymbol{\alpha} \sim F_{N,T-N-K}$$

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Test Statistic

- Define moments to estimate the coefficients $b' = [\alpha' \ \ \beta']$:

$$g_T(\boldsymbol{b}) = \begin{pmatrix} E_T(R_t^e - \alpha - \beta f_t) \\ E_T[(R_t^e - \alpha - \beta f_t)f_t] \end{pmatrix} = E_T \begin{pmatrix} \varepsilon_t \\ \varepsilon_t f_t \end{pmatrix} = 0.$$

- Define $\boldsymbol{d}$:

$$\begin{aligned} \boldsymbol{d} \equiv \frac{\partial g_T(\boldsymbol{b})'}{\partial b} &= -\begin{pmatrix} I_N & I_N E_T(f_t) \\ I_N E_T(f_t) & I_N E_T(f_t^2) \end{pmatrix} \\ &= -\begin{pmatrix} 1 & E_T(f_t) \\ E_T(f_t) & E_T(f_t^2) \end{pmatrix} \otimes I_N. \end{aligned}$$

Test Statistic

- Define S:

$$S = \sum_{j=-\infty}^{j=+\infty} \begin{pmatrix} E(\varepsilon_t \varepsilon_{t-j}') & E(\varepsilon_t \varepsilon_t' f_{t-j}') \\ E(F_t \varepsilon_t \varepsilon_{t-j}') & E(F_t \varepsilon_t \varepsilon_{t-j}' f_{t-j}') \end{pmatrix}.$$

- Using $a = I$ and the GMM variance formula:

$$var \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} = \frac{1}{T} d^{-1} S d^{-1'}.$$

Test Statistic

- Special case: assume that $\boldsymbol{f}$ and ε are independent, orthogonal and homoskedastic.
  - S simplifies to:

$$S = \begin{pmatrix} E(\varepsilon_t \varepsilon'_{t-j}) & E(\varepsilon_t \varepsilon'_t f_t) \\ E(f_t \varepsilon_t \varepsilon'_t) & E(f_t \varepsilon_t \varepsilon'_t f_t) \end{pmatrix} = \begin{pmatrix} 1 & E(f_t) \\ E(f_t) & E(f_t^2) \end{pmatrix} \otimes \Sigma.$$

Test Statistic

- Use $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ and $(A \otimes B)(C \otimes D) = AC \otimes BD$:

$$var \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} = \frac{1}{T} \begin{pmatrix} 1 & E(f_t) \\ E(f_t) & E(f_t^2) \end{pmatrix}^{-1} \otimes \Sigma = \frac{1}{Tvar(f)} \begin{pmatrix} E(f_t^2) & -E(f_t) \\ -E(f_t) & 1 \end{pmatrix} \otimes \Sigma.$$

- As a result:

$$var(\hat{\alpha}) = \frac{1}{T}(1 + \frac{E(f)^2}{var(f)})\Sigma.$$

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Cross-sectional Series Regressions

Result

We can test the model

$$E[R^{ei}] = \boldsymbol{\beta}_i' \boldsymbol{\lambda} + \alpha_i, i = 1, \ldots, N$$

by running cross-sectional regressions of average returns on the betas.

Cross-sectional Series Regressions

- start again with the K factor model:

$$E[R^{e,i}] = \boldsymbol{\beta}_i' \boldsymbol{\lambda}, i = 1, 2, \ldots, N$$

1. compute the betas in a time series regression

$$R_t^{e,i} = a_i + \boldsymbol{\beta}_i' \boldsymbol{f}_t + \varepsilon_t, t = 1, 2, \ldots, T$$

   for each i
2. compute the factor risk premia λ in a regression of average returns on betas

$$E_T(R_t^{e,i}) = \boldsymbol{\beta}_i' \boldsymbol{\lambda} + \alpha_i, i = 1, 2, \ldots, N$$

2 stages

1. First find estimates of the betas from time-series regressions:

$$R_t^{e,i} = \delta_i + \boldsymbol{\beta}_i' \boldsymbol{f}_t + \varepsilon_t^i \;\; t = 1, 2, ..., T \textit{for each } i.$$

2. Then, estimate the factor risk premia $\boldsymbol{\lambda}$ from a regression across assets of average returns on the betas:

$$E_T[R^{e,i}] = \boldsymbol{\beta}_i' \boldsymbol{\lambda} + \alpha_i \;\; i = 1, 2, ..., N.$$

OLS

- the OLS cross-section estimates:

$$\begin{aligned}\widehat{\boldsymbol{\lambda}} &= \left(\boldsymbol{\beta}'\boldsymbol{\beta}\right)^{-1}\boldsymbol{\beta}' E_T[\boldsymbol{R}^e] \\ \alpha &= E_T[\boldsymbol{R}^e] - \widehat{\boldsymbol{\lambda}}\boldsymbol{\beta}\end{aligned}$$

- if we assume errors are i.i.d. and independent of factors, we can just apply standard OLS machinery

Standard Errors

- we use Σ to denote the variance-covariance matrix of the errors:

$$\Sigma = cov(\varepsilon_t, \varepsilon_t')$$

- using OLS conventional formulas:

$$\begin{aligned}\sigma^2(\widehat{\boldsymbol{\lambda}_{OLS}}) &= \frac{1}{T}[(\boldsymbol{\beta}'\boldsymbol{\beta})^{-1}\boldsymbol{\beta}'\Sigma\boldsymbol{\beta}(\boldsymbol{\beta}'\boldsymbol{\beta})^{-1}] \\ cov(\widehat{\boldsymbol{\alpha}_{OLS}}) &= \frac{1}{T}[I - \boldsymbol{\beta}(\boldsymbol{\beta}'\boldsymbol{\beta})^{-1}\boldsymbol{\beta}']\Sigma[I - \boldsymbol{\beta}(\boldsymbol{\beta}'\boldsymbol{\beta})^{-1}\boldsymbol{\beta}']',\end{aligned}$$

where $\Sigma_f = cov(\boldsymbol{f}_t, \boldsymbol{f}_t)$ and $\Sigma = cov(\boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}_t)$.

Shanken correction

- betas are estimated!
- standard errors are corrected:

$$\sigma^2(\widehat{\boldsymbol{\lambda}_{OLS}}) = \frac{1}{T}[(\boldsymbol{\beta}'\boldsymbol{\beta})^{-1}\boldsymbol{\beta}'\Sigma\boldsymbol{\beta}(\boldsymbol{\beta}'\boldsymbol{\beta})^{-1}(1+\boldsymbol{\lambda}'\Sigma_F\boldsymbol{\lambda})+\Sigma_F]$$

$$cov(\widehat{\boldsymbol{\alpha}_{OLS}}) = \frac{1}{T}[I-\boldsymbol{\beta}(\boldsymbol{\beta}'\boldsymbol{\beta})^{-1}\boldsymbol{\beta}']\Sigma[I-\boldsymbol{\beta}(\boldsymbol{\beta}'\boldsymbol{\beta})^{-1}\boldsymbol{\beta}']'(1+\boldsymbol{\lambda}'\Sigma_F\boldsymbol{\lambda}),$$

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Fama-MacBeth

1. First find estimates of the betas from time-series regressions:

$$R_t^{e,i} = \alpha_i + \boldsymbol{\beta}_i' \boldsymbol{f}_t + \varepsilon_t^i \quad t = 1, 2, ..., T \ \text{ for each i.}$$

2. Second, run a cross-sectional regression *at each time period*:

$$R_t^{e,i} = \boldsymbol{\beta}_i' \boldsymbol{\lambda}_t + \alpha_{i,t} \quad i = 1, 2, ..., N \ \text{ for each t.}$$

3. Estimate $\boldsymbol{\lambda}$ and $\boldsymbol{\alpha}$ as the average of cross-sectional regression estimates:

$$\widehat{\boldsymbol{\lambda}} = \frac{1}{T} \sum_{t=1}^{T} \widehat{\boldsymbol{\lambda}}_t$$

$$\widehat{\alpha}_i = \frac{1}{T} \sum_{t=1}^{T} \hat{\alpha_{i,t}}.$$

Fama-MacBeth Standard Errors

- Use standard deviations of the cross-sectional regression estimates to generate the sampling errors:

$$\sigma^2(\widehat{\boldsymbol{\lambda}}) = \frac{1}{T^2} \sum_{t=1}^{T} (\widehat{\boldsymbol{\lambda}}_t - \widehat{\boldsymbol{\lambda}})^2$$

$$\sigma^2(\widehat{\alpha}_i) = \frac{1}{T^2} \sum_{t=1}^{T} (\widehat{\alpha_{i,t}} - \widehat{\alpha}_i)^2.$$

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Conditional Consumption CAPM

- consider the conditional Consumption-CAPM: (conditional linear factor model)

$$M_{t+1} = a_t - b_t \Delta c_{t+1}$$

- assume linearity:

$$a_t = \gamma_0 + \gamma_1 z_t$$

- assume linearity:

$$b_t = \eta_0 + \eta_1 z_t$$

Conditional Consumption CAPM

- substitute back into expression for M_{t+1}

$$M_{t+1} = \gamma_0 + \gamma_1 z_t - (\eta_0 + \eta_1 z_t) \Delta c_{t+1}$$

- scaled multi-factor model with constant coefficients
- the vector of factors:

$$F_{t+1} = (1, z_t, \Delta c_{t+1}, z_t \Delta c_{t+1})$$

and

$$M_{t+1} = c' F_{t+1}$$

- Lettau and Ludvigson propose cw_t as instrument z_t

Unconditional Multi-factor Beta Representation

Result

$$E[R_{t+1}] = \gamma_0 + \lambda' \beta.$$

where β^i is a vector of regression coefficients of of returns on the multiple factors.

- not a two factor beta model, because there is a cross-term!

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Conditional Human Capital CAPM

- consider the conditional Consumption-CAPM: (conditional linear factor model)

$$M_{t+1} = a_t - b_t R_{w,t+1} - c_t \Delta y_{t+1}$$

- assume linearity:

$$a_t = \gamma_0 + \gamma_1 z_t$$

- assume linearity:

$$b_t = \eta_0 + \eta_1 z_t$$

Conditional Human Capital CAPM

- substitute back into expression for M_{t+1}
- scaled multi-factor model with constant coefficients
- the vector of factors:

$$F_{t+1} = (1, z_t, \Delta y_{t+1}, z_t \Delta y_{t+1}, R_{w,t+1}, z_t R_{w,t+1})$$

and

$$M_{t+1} = c' F_{t+1}$$

- Lettau and Ludvigson propose cw_t as instrument z_t

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Conditional Human Capital CAPM

- time varying betas seem to help explain asset pricing puzzles
- Lewellen and Nagel measure conditional betas directly by estimating the CAPM on rolling windows

Jagannathan and Wang

- start with conditional CAPM:

$$E_t(R^i_{t+1}) = \gamma_{0,t} + \lambda_t' \beta^i_t.$$

Result

The conditional CAPM implies that the unconditional expected return equals:

$$E(R^i) = \overline{\gamma}_0 + \overline{\lambda}\overline{\beta}^i + Cov\left(\lambda_t, \beta^i_t\right),$$

with $\overline{\gamma}_0 = E[\gamma_{0,t}]$ *and* $\overline{\lambda} = E[\lambda_t]$ *and* $\overline{\beta}^i = E[\beta^i_t]$

- note: the expected β is not the unconditional β

Conditional CAPM

- the average pricing error

$$\alpha = E(R^i) - \beta^i \lambda - \gamma_0.$$

with β^i denoting the unconditional β and λ denoting the unconditional market price of risk.

- after substituting for $E(R^i)$, we get the average pricing error:

$$\alpha = (\overline{\beta}^i - \beta^i)\lambda + cov(\beta_t, \lambda_t)$$

- can α be large enough? depends on size of $cov(\beta_t, \lambda_t)$

Rolling Windows

- start with conditional CAPM:

$$R_{i,t} = \alpha_i + \beta_i R_{w,t} + \varepsilon_{i,t}$$

- directly estimate betas and alphas using short-window regressions: direct estimate of conditional alphas and betas without using any state variables
- check whether average conditional alphas are zero
- they're not