Mgmt 239c: Problem Set 1 *

Hanno Lustig

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This first problem set is due Friday, January 23. You can work in groups of 4. It is sufficient to hand in one problem set per group. You are encouraged to submit your matlab code together with the write-up of your answers. This problem set examines the LRR model.

1. *LRR Model Part I*: Consider the aggregate consumption growth process specification of Bansal and Yaron (2004), henceforth BY:

$$\Delta c_{t+1} = \mu + x_t + \sigma_t \eta_{t+1}, \tag{1}$$

$$x_{t+1} = \rho_x x_t + \varphi_e \sigma_t e_{t+1}, \tag{2}$$

$$\sigma^2_{t+1} = \sigma^2 + \nu_1(\sigma^2_t - \overline{\sigma}^2) + w_{t+1}, \tag{3}$$

where (η_t, e_t, w_t) are i.i.d. mean-zero, variance-one innovations. Consumption growth contains a low-frequency component x_t and is heteroscedastic, with conditional variance σ^2_t. These two state variables capture time-varying growth rates and time-varying economic uncertainty.

Preferences The long-run risk literature works off the class of preferences due to Kreps and Porteus (1978) and Epstein and Zin (1989, 1991). The equilibrium SDF can be stated as a function of aggregate consumption growth and the market return:

$$M_{t+1} = \beta^{\frac{1-\alpha}{1-\rho}} \left(\frac{C_{t+1}}{C_t}\right)^{-\rho\frac{1-\alpha}{1-\rho}} R_{t+1}^{\frac{\rho-\alpha}{1-\rho}} \tag{4}$$

You can proceed under the assumption that the log returns are generated exactly by this equation:

$$r_{t+1} = \kappa_0 + \Delta c_{t+1} + wc_{t+1} - \kappa_1 wc_t.$$

See the appendix for a derivation of the linearization coefficients.

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(a) **Wealth/Consumption Ratio**: We conjecture that the log wealth-consumption ratio is linear in the two states x_t and $\sigma_t^2 - \bar{\sigma}^2$,

$$wc_t = A_0 + A_1 x_t + A_2 \left(\sigma_t^2 - \overline{\sigma}^2\right).$$

Verify this conjecture from the Euler equation

$$E_t[e^{m_{t+1}+r_{t+1}}] = 1 \Leftrightarrow E_t\left[m_{t+1}\right] + E_t\left[r_{t+1}\right] + \frac{1}{2}V_t\left[m_{t+1}\right] + \frac{1}{2}V_t\left[r_{t+1}\right] + Cov_t\left[m_{t+1}, r_{t+1}\right] = 0 \quad (5)$$

Derive expressions for (A_0, A_1, A_2). Following BY, we assume joint conditional normality of consumption growth, x, and the variance of consumption growth. Use the following notation to simplify the analysis: $\lambda_{m,\eta} = -\alpha$, $\lambda_{m,e} = \frac{\alpha-\rho}{1-\rho}B$, $\lambda_{m,w} = \frac{\alpha-\rho}{1-\rho}A_2$, and $B = A_1\varphi_e$.

(b) **Calibration**: We calibrate the long-run risk model choosing the benchmark parameter values of Bansal and Yaron (2004). Since their model is calibrated at monthly frequency but the data are quarterly, we work with a quarterly calibration instead. We use $\rho = 2/3$, $\alpha = 10$, and $\beta = .996958$ for preferences and $\mu = .0045$, $\overline{\sigma} = .013531$, $\rho_x = .938314$, $\varphi_e = .126371$, $\nu_1 = .961505$, and $\sigma_w = .39324 * 10^{-5}$ for the cash-flow processes in (1)-(3). Solve the (non-linear) system of equations and find the loadings of the state variables in the log wealth-consumption ratio expression .

(c) **Risk Premium**: Derive an expression for the risk premium on the consumption claim $E_t\left[r_{t+1}^e\right]$ and the risk-free rate r_t^f. Evaluate at the calibrated parameter values.

(d) **CS Decomposition**: Derive the Campbell-Shiller Decomposition for the wealth/consumption ratio (analytically):

$$wc_t = \frac{\kappa_0}{\kappa_1 - 1} + \Delta c_t^H - \Delta r_t^H$$

Evaluate at the calibrated parameter values.

(e) **Currency**: Consider a symmetric 2-country version. Assume markets are complete. Derive the expression for the log changes in the exchange rate. Assume that the innovations to x are perfectly correlated and assume that the temporary consumption innovations have a correlation of .3. All other innovations are uncorrelated. Simulate to compute the volatility of the changes in the real exchange rate.

2. *LRR Part II*: Price a claim to aggregate dividends, where the dividend process follows the

specification in Bansal and Yaron (2004):

$$\Delta d_{t+1} = \mu_d + \phi x_t + \varphi_d \sigma_t u_{t+1} \tag{6}$$

The shock u_t is orthogonal to (η, e, w). This specification does not impose cointegration between consumption and dividends. You can assume the log return on a claim to the aggregate dividend is exactly equal to:

$$r^m_{t+1} = \Delta d_{t+1} + pd_{t+1} + \kappa^m_0 - \kappa^m_1 pd_t,$$

with coefficients

$$\kappa^m_1 = \frac{e^{A^m_0}}{e^{A^m_0} - 1} > 1, \text{ and } \kappa^m_0 = -\log\left(e^{A^m_0} - 1\right) + \frac{e^{A^m_0}}{e^{A^m_0} - 1} A^m_0$$

which depend on the long-run log price-dividend ratio A^m_0. We denote the return on financial wealth by a superscript m.

(a) **Price/Dividend Ratio**: We conjecture, as we did for the wealth-consumption ratio, that the log price dividend ratio is linear in the two state variables:

$$pd_t = A^m_0 + A^m_1 x_t + A^m_2 (\sigma^2_t - \bar{\sigma}^2).$$

Solve for the loadings following the same steps as before. Please use the following notation: $\beta_{m,e} = A^m_1 \varphi_e$ and $\beta_{m,w} = A^m_2$.

(b) **Equity Risk Premium**: Derive the equity risk premium on the dividend claim $E_t\left[r^{e,m}_{t+1}\right]$.

(c) **CS Decomposition**: Derive the Campbell-Shiller Decomposition for the dividend claim.

References

Bansal, R., and A. Yaron (2004): "Risks for the Long Run: A Potential Resolution of Asset Prizing Puzzles," *The Journal of Finance*, 59, 1481–1509.

Campbell, J. Y. (1991): "A Variance Decomposition for Stock Returns," *Economic Journal*, 101, 157–179.

Epstein, L. G., and S. Zin (1989): "Substitution, Risk Aversion and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," *Econometrica*, 57, 937–969.

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Kreps, D., and E. L. Porteus (1978): "Temporal Resolution of Uncertainty and Dynamic Choice Theory," *Econometrica*, 46, 185–200.

A Wealth/Consumption Ratio

For consistency with the timing of the budget constraint in the models that are to follow, all returns in this paper are ex-dividend instead of cum-dividend. The return on a claim to aggregate consumption, the *total wealth return*, is defined as

$$R_{t+1} = \frac{W_{t+1}}{W_t - C_t} = \frac{C_{t+1}}{C_t}\frac{WC_{t+1}}{WC_t - 1}.$$

In what follows, we use lower-case letters to denote natural logarithms. The notation wc_t denotes the log wealth-consumption ratio

$$wc_t = w_t - c_t = \log\left(\frac{W_t}{C_t}\right),$$

where wealth is measured at the beginning of the period and the consumption flow is over the ensuing period. Likewise $cw_t = -wc_t$ is the log consumption-wealth ratio. We start by using the Campbell (1991) approximation of the log total wealth return around the long-run average log wealth-consumption ratio A_0. Because the return is ex-dividend, this log-linearization delivers:

$$r_{t+1} = \Delta c_{t+1} + wc_{t+1} + \kappa_0 - \kappa_1 wc_t, \tag{7}$$

with linearization coefficients that are non-linear functions of the long-run log wealth-consumption ratio A_0

$$\kappa_1 = \frac{e^{A_0}}{e^{A_0} - 1} > 1 \text{ and } \kappa_0 = -\log\left(e^{A_0} - 1\right) + \frac{e^{A_0}}{e^{A_0} - 1}A_0. \tag{8}$$

By iterating forward on equation (7), we arrive at an expression that links the log consumption-wealth ratio at time t to expected future total wealth returns and consumption growth rates:

$$wc_t = \frac{\kappa_0}{\kappa_1 - 1} + E_t\left[\sum_{t=1}^{H} \kappa_1^{-j}\Delta c_{t+j}\right] - E_t\left[\sum_{t=1}^{H} \kappa_1^{-j} r_{t+j}\right] + E_t\left[\kappa_1^{-H} wc_{t+H}\right]. \tag{9}$$

Because this expression holds both ex-ante and ex-post, one is allowed to add the expectation sign on the right-hand side. Imposing the transversality condition as $H \to \infty$ drops the last term, and delivers the familiar Campbell-Shiller decomposition of the "price-dividend" ratio for the *consumption claim*, the wealth-consumption ratio:

$$wc_t = \frac{\kappa_0}{\kappa_1 - 1} + E_t\left[\sum_{t=1}^{H} \kappa_1^{-j} \Delta c_{t+j}\right] - E_t\left[\sum_{t=1}^{H} \kappa_1^{-j} r_{t+j}\right] = \frac{\kappa_0}{\kappa_1 - 1} + \Delta c_t^H - r_t^H. \tag{10}$$

We denote the cash-flow component by Δc_t^H and the discount rate component by r_t^H.